New Paradigms and Mathematical Methods for Complex Systems in Behavioral Economics

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Contents

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Abstract

This dissertation is devoted to the mathematical investigation of properties of complex socio-economic systems, where individual behaviors, and their interactions, exert a crucial influence on the overall dynamics of the whole system.

In order to understand the importance of such an investigation, it is necessary to briefly analyze some conceptual aspects relating to the interaction between applied mathematics and socio-economic sciences. The main issue in this field consists in coupling the usual qualitative interpretation of socio-economic phenomena with an innovative quantitative description by means of mathematical equations. This dialogue, however difficult, is necessary to reach a deeper understanding of socio-economic phenomena, where deterministic rules may be stochastically perturbed by individual behaviors. The difficulty mostly stems from the fact that the behavior of socio-economic systems, where the collective dynamics differ from the sum of the individual behaviors, is a paradigmatic example of a complex system. These aspects are discussed in the introductory section that follows.

The mathematical framework presented in this dissertation is built by suitable developments of the so-called mathematical kinetic theory for active particles, which proved to be a useful reference for applications in many fields of life sciences. The description of a system by the methods of the mathematical kinetic theory essentially implies the definition of the microscopic state space of the interacting entities and of the distribution function over this state space.

In the case of living systems, the identification of the microscopic state space requires the definition of an additional variable, called activity, which captures the specific dynamical aspects of the system under consideration. Entities of living systems, called active particles, may be organized into several interacting populations.

This dissertation presents in a unified context the results of the doctoral work, mostly described in four peer-reviewed research papers that are included as appendices of this dissertation. The essential ideas of each of the papers are introduced and summarized next. The first three papers exploit tools and developments of the kinetic theory for active particles, while the fourth paper is based on a different tool, namely on agents' methods.

The first paper, [3], develops a mathematical framework based on the kinetic theory for active particles, which describes the evolution of large systems of interacting entities. These entities are carriers of specific functions, in this case economic activities. The mathematical framework is constructed by means of a suitable decomposition into functional subsystems, namely aggregations of entities, which have the ability of expressing socio-economic purposes and functions.

The paper shows how this framework can be implemented to describe some specific complex economic applications. Specifically, the applications are focused on opinion dynamics and job mobility phenomena. These two examples offer a first insight into multi-scale issues: starting from the application, a preliminary mathematical framework which takes into account both microscopic and macroscopic interactions is developed. This framework may be adapted to a great variety of complex phenomena.

The second paper, [4], contains the initial elements of the development of the mathematical theory for complex socio-economic systems, already introduced in the first paper. The approach is based on the methods of the mathematical kinetic theory for active particles. The key concept of functional subsystem is analyzed in detail, developing suitable mathematical models, which involve the decomposition of the overall system into functional subsystems. Different examples of socio-economic phenomena are taken into account, in order to provide an application background. The theoretical framework is then adapted to a specific application, dealing with opinion formation dynamics, which leads to numerical simulations, that show some preliminary interesting results.

The third paper, [5], further develops the theory introduced in the first and second papers, with the setting of a mathematical model, where external actions play a key role. The aim of [5] consists in showing the emergence of collective behaviors or macroscopic trends from interactions at the microscopic scale, where agents are grouped into functional subsystems. The approach is, again, based on the methods of the mathematical kinetic theory for active particles: in this application the specific functions expressed by the interacting entities are socio-political activities subjected to the influence of media. Also in this case, numerical simulations show a direct application of the theoretical model, by means of specific settings of key parameters of the model.

Finally, the fourth paper, [6], derives an agent-based model, which allows the investigation of the socio-economic phenomenon of fashion. The model introduces two classes of agents, common agents and trend-setters, which play the dynamics that rule the emerging behaviors investigated by means of numerical simulations. Both the numerical and analytical tools used in this last paper differentiate it from the previous three works. Nevertheless, except for some conceptual differences, the approach is still focused on extracting emerging behaviors from individual-based interactions. The goal is to illustrate and implement different methodologies within the same research environment.

Chapter 1

Introduction

1.1 Socio-economic motivations

This thesis is devoted to the investigation of the complex mechanisms which rule the transition from the behavior of single individuals to the collective behavior of groups of people, in social phenomena. The dynamics of our societies are ruled by many complex socio-economic phenomena, which still lack a proper support with refined mathematical models. A deeper understanding would lead to important improvements in the explanation of various events of our times.

From a historical point of view, social sciences are relatively young, and, as a consequence, their formalization is still at the beginning. Important contributions to the modeling of classical scientific phenomena related to disciplines such as physics and engineering have been developed in the last centuries. Therefore that mathematical models which describe natural and technological processes have reached a level of complexity, often sufficient to describe and forecast quite precisely the outcomes of the phenomena they take into account. On the contrary, the mathematical modeling of social sciences still needs a proper setting. Despite many fields of economics and related disciplines (such as political economy, industrial organization and so on) nowadays commonly exploit advanced mathematical modeling, many fundamental aspects of socio-economic processes still have to be clarified.

Moreover, economic phenomena always involve decisions of human beings, both when studying the organization of a small firm and when dealing with an extremely complex macroeconomic issue. As a consequence, thanks to the peculiarity of dealing with entities involving human beings, it is impossible not to consider the fact that people do not always follow rules that remain constant over time, and large scale experiments are almost impossible. This is the reason why modeling human behaviors is extremely complicated, and requires very general mathematical paradigms to be defined, which can be adapted and modified in order to deal with the specific aspects of each single application.

In addition, real socio-economic behaviors are so complex that they are hardly conformable to the concept of perfect rational agents. Many economic models, indeed, assume that people are hyper-rational entities: once their preferences are defined, they behave in a perfect rational way, doing nothing that violates these preferences, in order to obtain the highest possible well-being for themselves. The mathematical formalization of the concept of rational agents and preferences resorts to the definition of a utility function, which is optimized, namely maximized or minimized according to the specific model under consideration, given the constraints which represent the practical conditions, which restrict the free behavior of the agent. That is, the individual seeks to attain very specific and predetermined goals to the greatest possible extent with the least possible cost. In economics, and in all social sciences in general, these assumptions must be considered as approximations to real social behavior; in facts, sociologists tend to prefer more structural explanations to the ones based on the concept of rational agent.

In particular, in the recent scientific literature, many critics to the rational agent approach have appeared, introducing the concept of *bounded rationalities*. This term is credited to Herbert Simon, [76], [78], [79]. Simon believed that agents are only partly rational, being emotional and irrational in the most part of their actions; as a consequence, agents cannot make fully rational decisions, possessing just what he called bounded rationality. He also paid attention to the difficulties in obtaining information or in problem solving, which are essential features of human life: *boundedly rational agents experience limits in formulating and solving complex problems and in processing (receiving, storing, retrieving, transmitting) information* (Williamson, p. 553, citing Simon). For all these reasons, he simply suggests that agents use behavioral heuristic, rather than a rigid rule of optimization, when taking decisions. A peculiar feature of his theory is the attention devoted to authority, defined as the ability and right of an individual of higher rank to determine the decision of an individual of lower rank. Authority is considered to be highly influential on the global structure of decision making, thanks to patterns of communication, sanctions and rewards, as well as on the establishment of goals, objectives, and values of the organization.

Another fundamental critic to rational agent models is proposed by Ariel Rubinstein in [75]: he suggests to model bounded rationalities by explicitly specifying decision making procedures. This puts the study of decision procedures on the research agenda. As the author writes in the introduction of this book: *usually economic models do not spell out the procedures by which decisions of the economic units are made; here we are interested in models in which procedural aspects of decision making are explicitly included.* Gerd Gigerenzer [56] is a German psychologist who has deeply studied bounded rationalities proposing to examine the use of heuristics in decision making, showing that often simple heuristics lead to better decisions than the theoretically optimal procedure. He conceives bounded rationality as an adaptive tool, different from formal logic or probability calculus.

Brian Arthur and the Santa Fe School have had a key role in this debate, with similar ideas, although with highly different quantitative approach. This branch of the literature pays attention to the definition of *complex dynamic systems*, considering economic processes as typical example of emerging dynamic phenomena. Quoting [12], Bertrand Russel dropped interest in economics because it was too simple, Max Planck dropped his interest because it was to difficult. As Brian Arthur wisely suggests, it can be simple or difficult depending on how economic problems are formulated. If the assumption of a rational agent who makes rational decisions is made, generally a well defined solution or equilibrium follows. But how agents get from the problem to the solution is a *black box*. When we try to open this box, suddenly economics becomes extremely difficult. Indeed, between the problem and the solution lie all the cognition and decision processes undertaken by agents, that make economics so difficult. According to the Santa Fe school, it is necessary to stress and pay more attention to the origin of the cognitive and experience processes, which induce social economic behavior. In [13] the idea of indeterminate problems in economics caused by the differences among different agents is developed. Crashes and bubbles are the so called empirical anomalies that remain unexplained phenomena and are not predictable.

According to [14] a remarkable similarity exists among all complexity studies: all elements and patterns vary from one context to another. As the agent reacts, the aggregate elements react anew: complex systems are system in process, that constantly evolve. Systems of this kind are quite common in economics: economic agents, like banks, consumers, firms, investors, behave and determine the market moves, acting together. Moreover, unlike inert matter, human agents react with strategy and foresight, considering outcomes that result as a consequence of behavior they might undertake. This adds a layer of complications to all social sciences, and to economics in particular. Conventional economic models try to simplify this unfolding patterns, seeking analytical solutions. Nowadays, instead, a large branch of economists are turning to the analysis of how actions, strategies, and expectations, might interact to create the aggregate emergent behaviors. The result is a theory at a more general out-of-equilibrium level, involving systems that generally do contain non linearities. All these concepts have been formalized and ordered in the introduction of [15], namely the Proceedings of an August 1996 workshop sponsored by the Santa Fe Institute Economics Program, apt to give a formal insight to the new approach of *Economic as an Evolving Complex System*. In this volume they present a very precise list of the features of complexity:

• **Dispersed interactions**: phenomena are characterized by the interaction of many dispersed agents interacting in parallel. The action of each agent depends upon the anticipated actions of a limited number of other agents and on the aggregate state these agents co-create.

• **No global controller**: no global entity controls interactions. Economic actions are mediated by legal institutions, assigned roles, and shifting associations.

• **Cross-cutting hierarchical organization**: the economy has many levels of organization and interaction. Units at any given level, with their behaviors, actions, and strategies, serve as building blocks for constructing units at the next higher level. The overall organization is more than hierarchical, with many sorts of tangling interactions across levels.

• **Continual adaptation**: behaviors, actions and strategies are revised continually, as individual agents accumulate experience: the system constantly adapts.

• **Perpetual novelty**: novelty niches are continually created by new markets, new technologies, new behaviors, new institutions.

• **Out of equilibrium dynamics**: new possibilities are continually created: the economy operates far from any optimal equilibrium.

Let us stress that these features are very common in most social and natural environments: nervous systems, ecologies, biological systems and economies. The shift from a static outlook into a process orientation is common to all complexity studies. In general, economic patterns sometimes simplify into the simple homogeneous equilibria of standard economics; more often they are ever changing, showing perpetually novel behavior and emergent phenomena. Complexity portrays the economy not as a deterministic predictable mechanical system, but as process-dependent, organic, and always evolving. Notwithstanding neoclassical economics, agents learn, adapt, use cognitive processes. Agents interpret the world they inhabit which is complicated by the presence of other interacting agents. Agents generally do not optimize, because the concept of the optimal course of action cannot be defined. Agents learn from each other, and they can be influenced by individual strategies and actions. Because of these complexity features, mathematical tools such as linearity, or fixed points, cannot lead to a deep understanding of the system. A new class of combinatorial mathematics and population level stochastic processes, in conjunction with computer modeling, are required. These tools emphasize the discovery of structures, and the process through which the structure emerges across different levels of organization.

Let us give a famous example, [16], derived from everyday life: *the El Farol Problem*. N people decide independently each week whether to go to a well known bar in Santa Fe, whose name is El Farol, that offers entertainment on a certain night. Space is limited, and the bar is enjoyable if it is not too crowded (fewer than 60 people go). There is no way to tell for sure the number of people coming in advance. An agent goes if he expects fewer than 60 to go, but no communication takes place among agents: the only information available is the number who went last week. If all believe that few will go, all will go. If all believe that most will go, nobody will go. Starting from this very simple idea, a dynamic mathematical model is constructed, followed by computer experiments, which show which kind of dynamic and emergent behavior takes place.

Another important researcher in the field is Duncan J. Watts, an Australian professor of sociology at Columbia University, where he heads the Collective Dynamics Group. In his research, he explores the *role that network structure plays in determining or constraining system behavior, focusing on a few broad problem areas in social science such as information contagion, financial risk management, and organizational design.*(D. J. Watts home page). In [82], Watts and Strogatz presented a mathematical theory of the *small world phenomenon*. The Watts and Strogatz model is a random graph generation model that produces graphs with small-world properties, namely short average path lengths and high clustering. The small world phenomenon was first investigated by Stanley Milgram [66], who conducted several experiments devoted to examine the average path length for social networks of people in the United States. The research suggested that human society is a small world type network characterized by short path lengths: Milgram's experiment was originated by the desire to learn more about the probability that two randomly selected people would know each other. The experiments are well known also with the phrase *six degrees of separation*, although Milgram did not use this term himself, because the research concluded that people in the United States are separated by about six hops on average. In mathematics and physics, a small-world network is a type of mathematical graph in which most nodes are not neighbors of one another, but most nodes can be reached from every other by a small number of hops. Social networks, the connectivity of the Internet, and gene networks, all exhibit small-world characteristics. Watts and Strogatz noted that graphs could be classified according to two independent structural features, namely the clustering coefficient and the average node-to-node distance, the latter also known as average shortest path length. Moreover, they measured that in fact many real-world networks have a small average shortest path length, but also a clustering coefficient significantly higher than expected by random chance.

Finally, another important work which proposes a new idea of modeling economics as social science is [62] and more in general [63], which again shows the importance of social networks and interactions in economics. In the first paper, the wholesale fish market in Marseille has been studied, introducing the idea of high loyalty of buyers to sellers, and persistent price dispersion taking place in the same place every day. The authors build an adaptive agent model, where sellers set quantities, prices and treatment to customers, buyers instead decide which seller to visit and at which price to buy. Then, the emerging behavior is analyzed. The second work, instead, is a collection of articles having in common the mathematical modeling approach to economics, taking into account *heterogeneous interacting agents*. In its introduction, the reconciliation among economics, psychology and sociology is highlighted, thanks to the new cognitive approach towards economics in general. All the economics literature which examines economic phenomena through social networks is much closer to what sociologist have been studying for years than standard contributions to economics. In facts, the link between the structure of the network, the interactions among agents, and the resulting outcome, is often difficult to understand: generally, the result depends on how the individuals are linked. This problem is again a standard feature of complexity, related to the transition *from micro to macro behavior*. Another key concept to understand properly this behavior is the tendency for agents to form coalitions, using strategies. Coalitions themselves become actors that influence the final aggregate behavior. Finally, interactions and knowledge (which themselves define the dynamic rules) are key parameters of the emerging pattern of the economic process.

This thesis places itself in this new approach to economic modeling, and distinguishes from standard economics literature. It proposes a critic to rational agent models, building up a new mathematical theory apt to model socio-economic phenomena, which takes into account all the aspects discussed in this chapter. Its goal is to provide a new contribution for the solution of the so-called black box problem in socio-economic sciences, trying to clarify the unknown bridge from the micro to the macro level. Probabilistic dynamical methods will be described, together with computer simulations related to some practical applications. The quantitative aspects and the related applications will be analyzed in details in the following chapters.

Let us finally provide some general guidelines that have been pursued in the research agenda:

• *Phenomenological and qualitative analysis of the systems under consideration and development of a modeling strategy related to derivation of mathematical tools.*

• *Derivation of specific models according to afore mentioned mathematical framework and tools.*

• *Statement of mathematical problems generated by the application of models.*

• *Simulations and critical analysis of models.*

The contents of the thesis are organized in two chapters, and an appendix which follow the above introduction.

Chapter 2 proposes a summary of the four peer reviewed papers present in the appendix.

Chapter 3 analyzes research perspectives.

The appendix reports the papers from which the concepts hereby described are taken from.

1.2 Introduction to the mathematical framework

This thesis deals with the development of a mathematical theory to model complex socio-economic systems, where individual behaviors and interactions play a significant role on the evolution of the system itself. The objective of developing a mathematical description of complex social-economic systems is a challenging, however difficult, perspective of interaction between mathematical and social sciences. The main objective consists in the assessment of the paradigms which can act as the conceptual background for a unified approach to the modeling of a variety of social systems characterized by common features.

In order to deeply understand the need of such a theory it is worth analyzing some conceptual aspects on the interaction between applied mathematics and social sciences. The main problem consists in investigating how the *qualitative interpretation* of social reality, that is delivered, including the interpretation of empirical data, by research activity in social sciences, can be transferred into a suitable *quantitative description* through mathematical equations. The above target needs the development of a dialogue between methods and traditions of two different disciplines with the additional difficulty of dealing with living systems, which have the ability to think and, as a consequence, to react to external actions, without following rules constant over time. This dialogue, however difficult, is necessary to a deeper understanding of the so called *behavioral economics*, where deterministic rules may be stochastically perturbed by individuals behaviors. An additional difficulty to be taken into account is that individual behaviors not only show random fluctuations, but may be substantially modified by external environments, from depressive or panic situations to over-optimistic attitudes induced by misleading propaganda.

Moreover, the behavior of a complex system, namely a system of several individuals interacting in a non-linear manner, is generally impossible to describe and model only by the dynamics of a limited number individual entities. The interest on this kind of systems has seen, in recent years, a remarkable increase, due to a raising awareness that many socio-economic systems sharing this kind of complexity cannot be successfully modeled by traditional methods used for other economic environments. Generally, these complex phenomena are such that the collective dynamics of the system under consideration is determined by complex collective interactions, which differs from the sum of the individual dynamics.

This program is pursued by suitable development of the so called mathematical kinetic theory for active particles [11],[20] KTAP, which has shown to be a useful reference applications in other fields of life sciences such as the immune competition [23], traffic flow [44], social dynamics [25], [26] psychological interactions [21], [37]. The description of the system by methods of the mathematical kinetic theory essentially means defining the microscopic state of the interacting entities and the distribution function over the above state. In the case of living systems, the identification of the microscopic state requires the definition of an additional variable, called activity, which captures the specificities of the system under consideration. Entities of living systems, called active particles, may be organized into several interacting populations. Stochastic games can be used to model interactions at the microscopic scale, while methods of the mathematical kinetic theory can be used to derive evolution equations for the probability distribution function over the microscopic state of the interacting active particles. Macroscopic quantities are obtained by weighted moments of the afore mentioned distribution function.

Several attempts to combine social economic sciences and mathematical modeling can be found in the literature: in particular the main branches are agent-based models, game-theory, population dynamics, and social networks. Let us briefly describe the main feature of each of them in order to understand in which sense they distinguish from the kinetic theory for active particles and in which cases they can provide an important a complement to this kind of modeling.

Agent-based models are computational models in which interactions among individuals (agents) are simulated, in order to understand the emerging behavior of the macroscopic overall system, starting from the microscopic dynamics of each agent. The goal is to find the equilibria or the emergent patterns in the complex system, as a consequence of the dynamics of the agents. This approach combines elements of game theory, multi agent systems and Monte Carlo methods. These models consist in systems of equations that simulate the simultaneous behavior of multiple agents, starting from specific sociological assumptions, in order to recreate and predict the global behavior of the system.

Agent based models can be considered a complement to traditional analytic methods, since they may recreate the equilibria or non-equilibria resulting from the analytic method itself. Indeed, they can explain the emergence of unpredictable patterns and distinguish among different types of network structures. These models have been recently used to describe a great variety of network-related phenomena of recent times, like internet, terrorism, traffic jams, financial crisis, consumer behavior, spread of epidemics and social segregation. Notice that all the phenomena listed above are strongly affected by individual or particles interaction and behavior. Some key references can be found in [50] and [17].

Game theory is a branch of applied mathematics that has become widely used in social sciences and economics. The goal of game theory is to describe behaviors of individuals (players), who designs specific strategies to give the best response to other players'choices. Game theory is widely applied in all the rational modeling of social-sciences. It aims at finding equilibria in the sets of strategies and different kinds of equilibria have been defined. Starting from the analysis of economic competition with the the 1944 book *Theory of Games and Economic Behavior* by John von Neumann and Oskar Morgenstern [68], game theory it is now widely used to model biological, social and even telecommunication interactions. Most notable references can be found in [73], [74] and [57].

Population dynamics, instead, studies how the number of individuals in one or several populations changes, under the action of biological and environmental processes. It focuses on the rise and decline of different populations of different particles: individuals, cells and so on. It has been a dominant branch of mathematical applications to biology for more that 200 years. In the early 19th century it widely investigated demographic studies, whose main modeling aspects are related to the following dynamic rate functions of the considered populations: natality or birth rate, growth rate and mortality. For a complete survey on the subject it is possible to refer to [81].

Particularly important is, in economic and social applications, the use of population dynamics with internal structure, where an additional internal variable can model specific characteristics of the system under consideration. Large systems of interacting individuals can be modeled by systems of partial differential equations taking into account internal variables describing the socio-economic state of the individuals. This internal structure describes the peculiarities of the system under consideration, and contributes to model the emergence of a collective behavior. An introduction to population dynamics with internal structure is given in [83],[46] and [69].

Notice that, in population dynamics, the internal structure is a deterministic variable, in the KTAP, instead, the internal variable is heterogeneously distributed among agents, while interactions modify such distribution. Indeed, this is an essential ingredient to model social systems in the framework of behavioral economics.

The study of *social networks* is receiving growing attention, focused on the role of networks in determining and constraining social behavior. Starting from the Milgram's experiment [66], and the celebrated paper developing the concept of *small-world phenomena* [82], it is possible to find, in the recent literature, many socio-economic phenomena studied by this approach [58], [49], [52]. In particular, [62] and [63] show the importance of social networks and interactions in economics.The wholesale fish market in Marseille has been studied in [62], introducing the idea of high loyalty of buyers to sellers, and persistent price dispersion taking place in the same place every day. The authors build an adaptive agent model, where sellers set quantities, prices and treatment to customers, buyers instead decide which seller to visit and at which price to buy. Subsequently, the emerging behavior is analyzed. The second work [63], instead, is a collection of articles having in common the mathematical modeling approach to economics, which takes into account *heterogeneous interacting agents*. In its introduction, the reconciliation among economics, psychology and sociology, is highlighted thanks to the new cognitive approach towards economics in general.

The borders among these disciplines show, in practical applications, various flexibilities, in the sense that tools and methods of each of them are used together for a better development of mathematical models. In particular, a recent field of investigation, which wisely combines elements of the three branches described above is the so called *behavioral economics*. Its principal scope is to merge properly psychology and economics, stressing the importance of human behavior and psychological attitude in the evolution of complex systems of individuals. Behavioral economics widely use experiments as a preliminary step of the modeling process. To obtain an idea of the wide applications and philosophical foundations of behavioral economics it is possible to refer to [67], [60], [70], [71], [33], [72], [35], [36], [45], [38]. Moreover [34] and [28] provide an insight into the links between behavioral economics and game theory, in order to adapt the second to the challenges proposed by the first, introducing the concept of *behavioral game theory*. Additional references can be found in [47] and [48].

It is plain that stochastic aspects should be an essential ingredient in the models derived within the framework of the behavioral economics, for this reason, the KTAP framework seem to be an appropriate tool to derive models regarding this kind of phenomena. Let us briefly summarize this theory. Social systems are characterized by a large number of particles representing the interacting socio-economic entities, called active particles, (consumers, firms, institutions and so on), whose state may be characterized by both mechanical variables and an activity variable representing the specific socio-economic feature of the system itself. While the activity variable describes the microscopic state of the system, the global state is described by the probability distribution function over the microscopic state whose weighted moments recover the macroscopic quantities of the system. Notice that thanks to the activity variable, the output of the interactions is a direct consequence of the strategic behavior which takes place among active particles. The activity variable, in the more general case, can be different from particle to particle. The evolution in time and space of the system is derived by a balance of the inflow and outflow of active particles in the elementary volume of the space, as a consequence of the dynamics based on strategic interactions. Subsequently, the state of the interacting active particles is described by random variables, while the output of interactions is again delivered by random features.

Notice that this theory borrows and extends mathematical tools from the four branches described above. As in agent-based models, models derived from the

KTAP may be characterized by multi-agent (or multi-particle) interactions defined as a consequence of assumptions regarding the sociological features of the system under consideration, whereas the mathematical tools used are more refined since they generally allow to obtain systems of partial differential equations to be solved with computational methods. With regard to game theory, KTAP models describe particle interactions thanks to the definition of stochastic games, which characterize the particle response after the encounter with other particles. As in game theoretical models, the evolution of the system is affected by the definition of these games, even if the overall mathematical framework leads to systems of equations that are not present in traditional game theoretical models. The KTAP distinguishes from population dynamics since it takes into account an internal variable stochastically defined among particles, which better represents the random feature of human behaviors: in the KTAP models, indeed, the socio-economic state of the system is described by a probability density function. Finally, since the KTAP models describe the evolution of systems as a consequence of particles interaction, a network structure may arise in the mathematical framework of the model, helping the description of the interactions within the system. Concepts belonging to all these four mathematical methods are, then, taken into account and updated to develop a new mathematical approach, based on the concept of functional subsystem.

Let us now proposes a personal insight into complex living systems, of which socio-economic systems are a subset, focused on the assessment of specific features that play a key role in the mathematical interpretation by models and, possibly, by a mathematical theory.

Let us now consider a large system of interacting particles that belong to physical systems whose dynamics is determined by their ability to dialogue among themselves and develop specific strategies. Typical examples are consumers in the market, cells of multicellular systems in biology, pedestrians in crowds, vehicles-driver on roads, or animals in swarms.

If the strategy changes according with the number of interacting particles, all these systems can be classified as complex systems according to the observation that the dynamics of a few entities does not generate straightforwardly the dynamics of the whole system. Consequently, emerging collective behaviors do not appear to be straightforwardly related to the individual dynamics.

The various systems which have been listed above belong to the living matter. Therefore, their dynamics should include the description of essential aspects of living systems such as ability to develop specific strategies, competition, evolution, as well as reproductive and/or destructive ability. Although these systems belong to different fields of life sciences some common features can be identified and, ultimately, it is reasonable querying if it is possible extracting some common features for all of them.

• *Complexity and decomposition into functional subsystems*

Complexity is the most challenging feature of socio-economic systems: a huge number of stochastic interactions may generate emerging patterns of strategic behaviors, which vary according to the number and on the socio-economic state of the interacting particles. As a consequence, the problem of a suitable decomposition, apt to reduce the complexity of the system arises, in order to properly classify and model interactions. This is the reason why the concept of functional subsystem is introduced: more in details, the global framework, which corresponds to a specific socio-economic phenomenon, can be hierarchically decomposed into *functional subsystems*.

Each functional subsystem expresses a specific social function, called *activity* which plays an active role in the evolution of the global phenomenon and which can be properly mathematically modeled. Notice that the activity can be regarded as a continuous or discrete variable, depending on the applications. Particles may refer, according to the selected scale, to individuals, institutions, firms, interests groups, companies or every constituent element of a socio-economic society. Those particles expressing the same activity function are then grouped into the same functional subsystem: so that the overall system is decomposed in many functional subsystems, each of them expressing different abilities and purposes. Notice that this decomposition reduces the number of interacting entities and, as a consequence, the complexity of the system. Moreover, this procedure may help the modeling aspects, since it allows to treat and model separately different groups of particles with different behaviors and different strategies.

The dynamics is then defined through interactions among functional subsystems. Notice that interactions undergo a double dynamics: an inner dynamics due to interactions among particles within the same functional subsystem and an outer dynamics due to interactions among different functional subsystems. This decomposition is a flexible approach that can be adapted to many particular investigation and varies according to the phenomenon under consideration.

• *Interactions.*

An additional key feature of complex systems is the network of interactions among the different components of the system itself: since interactions are determinant in the emergence of a collective strategic behavior, namely in the modification of the state or of the strategy of the interacting particles, the definitions of the type and structure of interactions become essential, in the modeling procedure. Notice that the mathematical modeling of interactions cannot disregard the concept of functional subsystem: it has to be clear which subsystems are interacting and under which conditions.

• ***Distance and topology.***

Another problem which arises in the mathematical modeling of living systems refers to how their topological and state distribution affects interactions. Subsystems and particles may interact when they are sufficiently close either in a geometrical sense or in a microscopic state sense. In the first case interactions take place according to the spacial localization of particles and subsystems, in the second case according to their socio-economic state distribution. As a consequence, it seems appropriate to introduce the concepts of geometrical and state distances: keeping in mind that each particle or subsystem expresses a specific activity, which represents its social state, and it is physically set in the geometrical space, it is possible to define a distance in the expressed social function and a distance in the spacial localization. Subsystems with a low state distance are supposed to be close in their tasks and purposes and consequently to interact more often than subsystems with a high state distance.

On the other hand, subsystems with a low geometrical distance may interact more easily than subsystems characterized by a far space localization. Therefore, the frequency and the quality of interactions varies according to a balance of this two distances. Notice that the rules with which subsystems and particles interact, may define a network structure which depends both on the topological and on the state distribution of the system, whose links represent the possible existing connections among functional subsystems and particles. Links can be deterministically or stochastically modeled, depending on the quality of interactions. Once a network topology is modeled, it becomes possible to define other concepts of distances related to the number of hops which separate subsystems or particles on the network itself.

• ***Stochastic Games.***

According to their purposes, functional subsystems play specific strategic games, whose outcomes depend on the topological and state distribution of the interacting subsystems. Since, generally, the strategy of particles belonging to the same functional subsystem is randomly distributed, the final outcome is not deterministically predictable, therefore it must be modeled, in the more general case, from a stochastic point of view; this is the reason why the game is said to be stochastic. Notice that strategies are not constant on time: each subsystem may change strategies according to its geometrical position and its distribution of state within the system at every time t. Moreover, while the modeling of simple phenomena takes into account binary interactions, for a great variety of models it is necessary to study interactions stochastically distributed among every functional subsystem, increasing the complexity of the system.

• ***Multi-scale aspects.***

Multi-scale aspects are probably one of the main and most fascinating features of complexity, since they have not been properly investigated yet in economics and social sciences. First of all, the proper scales have to be defined regarding the different phenomenon under consideration: the suitable scales to model individual interactions are not the suitable ones to model state building processes or companies and institutions network structures: as a consequence, the definition of the micro and the macro level becomes a key modeling aspect. Moreover, in many living systems, the dynamics at the microscopic scale of the active particles affects the dynamics at the higher scale of the functional subsystems and vice versa: decisions taken by individuals (the active particles) working in a firm (the functional subsystem) may influence the strategic behavior of the firm with institutions and other companies (other functional subsystems), while the competition among firms may influence the strategic behavior of individuals within the same firm, just to give a very generic example.

Let us now give a preliminary introduction to the mathematical framework used in the modeling procedures. The details can be found in the publications in the appendix of this thesis.

Complex systems are generally characterized by a large number of interacting entities, all of them expressing different functions and strategies. Therefore, in order to develop a suitable approach for the analysis of these systems it is necessary to develop methods which are able to partition or reduce the overall complexity. Several attempts and approaches for the reduction of complexity were presented in the literature, adopting a proper decomposition of the system, with different aims and perspectives: among others it is possible to refer to [77] and [10] in economics and social sciences, [40], [41], [42], [43] in computer science, [59], in biology, [31] and [45] in economics.

Simon and Ando, ([77]), investigated the dynamic behavior of linear systems, introducing the concept of nearly-completely decomposable matrices, namely *matrices whose elements, except within certain submatrices along the main diagonal, approach to zero in the limit*. They showed that in this kind of systems it is possible to aggregate variables and separate the analysis of the short-run dynamics from the one of the long-run dynamics; in the short run, the behavior of the system may be approximated and thus analyzed as if it was completely decomposable. In the long run, instead, this is no longer possible. Similar properties were investigated and proved in a later paper by Ando and Fisher ([10]) for nearly decomposable systems. Both these papers aim at showing that a sufficient degree of near decomposability always exists, given a certain standard of approximation. In [40], [42], [43] the other side of the decomposition problem is investigated, namely these works do not focus on how the system can be decomposed, but focus on when the decomposition may be done without incurring in misleading approximations of the interactions inherent in the system behavior. Courtois seeks to establish which standard of approximation may be guaranteed when an aggregation procedure is applied to a given nearly-completely decomposable system, by looking for lower and upper bounds of the error originated by the decomposition of the system. Since the criteria for reducing a model into sub-models are not easy to determine, Courtois reviews the conditions under which there exists a valid approximation. Furthermore, he analyzes some useful properties of complex systems, that must be taken into account in the process of decomposition. He notes that interactions between subsystems are not negligible, although very low: if they were, the system would be completely decomposable and, consequently, no longer complex. Given the existence of these interactions, Courtois aims at answering the following question: *How should the sub-models that are analyzed in isolation be built so as to minimize the error that occurs when the weak interactions between these subsystems are ignored?* In the already cited papers he gives an estimation of this error. In [31] and [45] the issue of *modularity* is discussed both from a conceptual and from a problem-solving perspective. As the authors state: *modularity is in fact a decomposition heuristic, through which a complex problem is decomposed into independent or quasi-independent sub-problems.* Hartwell ([59]) proposes a modular approach towards the investigation of biological processes, where different modules with different functions interact in the cellular bio-mechanics. The biological system is decomposed into interacting entities that are carriers of specific biological functions.

These modular (or nearly decomposable) approaches share some conceptual similarities, although the aims and the procedures of the decompositions are very different. The common idea is to look for a suitable and possible decomposition, apt at reducing the complexity of the system, able to capture its specific complex features. Starting from these approaches, a new mathematical framework of decomposition for socio-economic systems focused on the functions expressed by the entities composing the overall system is developed. Let us introduce the concept of socio-economic functional subsystem according to the following definition:

A **functional subsystem** *is an assembly of socio-economic agents with the ability to express collective specific strategies that play a fundamental role in the dynamics. More specifically, depending on the phenomenon under consideration, the socio-economic framework is decomposed into sub-groups, called functional subsystems, having the same socio-economic aim and expressing the same activity variable.*

Notice that even the decomposition into functional subsystems is strictly related to the modeling investigation process: depending on which features of the problem it is interesting to focus on, the functional subsystems are properly constructed. This framework analyzes the evolution of the socio-economic activity variable, as a result of interactions among the different subsystems of the global structure.

One of the advantages of this method is that it creates a bridge among both the microscopic and the macroscopic scale. The concept of functional subsystem lyes in a intermediate scale which provides the link between the microscopic level (which describes the physical state of each particle) and the macroscopic one (which describes the physical state of the overall system).
Hereby a mathematical framework regarding the concept of functional subsystem is provided. The overall state of each functional subsystem is described by the distribution function and it is viewed as a network of interacting functional subsystems. Their representation is based on the assumption that each of them has the ability to express a specific activity, identified by the variable $u \in \mathbb{R}$, where the value $u = 0$ separates the positive and negative valued *functions* expressed by each subsystem:

$$f_i = f_i(t, \mathbf{x}, u) : \qquad [0, T] \times D_x \times D_u \to \mathbb{R}_+, \tag{1.1}$$

where the subscript $i = 1, ..., n$ identifies the ith functional subsystem, $\mathbf{x}$ the geometrical microscopic state, u the activity variable. f_i is called generalized distribution function and $f_i(t, \mathbf{x}, u) d\mathbf{x} du$ represents the number of active particles which at time t are in the elementary volume of microscopic states $D_x \times D_u$. Notice that this representation corresponds to the fact that social systems are identified by the geometrical variable and the social state of their components, namely the microscopical state is not scalar but defined by the vector $(\mathbf{x}, u)$. These two variables are those that play a role in the stochastic strategy dynamics described in the previous section.

The assessment of the role of the space variable needs further reasoning. Indeed, this variable in several interesting systems of social and economic sciences does not play an important role, considering that the communication among active particles is not influenced by localization, while, in other cases, the space variable identifies the region where a functional subsystem is localized. In this latter case, $\mathbf{x}$ identifies the functional subsystem by its localization to be also related to its activity. Therefore, the representation is given by

$$f_i = f_i(t, u) : [0, T] \times D_u \to \mathbb{R}_+, i = 1, ..., n \tag{1.2}$$

The analysis proposed in what follows refers to the representation (1.2).

The activity variable u can be a discrete or a continuous variable depending on the phenomena under consideration and on the constraints of the modeling process. More in details, the socio-economic function described by means of the activity variable can contribute to the modeling of different kinds of discrete systems at the microscopic scale. In particular, continuous systems which are discretized in order to be measured, discrete systems due to the fact that the number of interacting agents is limited and continuous systems that are modeled by means of a discrete filter.

Regarding the first type of systems a well-known example is the electoral competition of a large country, where the socio-political state can assume every single value corresponding to the personal political opinions of individuals; during the voting process, however, opinions are forced to be discrete since every agent has to express a preference on a limited number of parties.

As far as the second class of systems is concerned, if we think, for instance, of a voting process in a local assembly, where the number of voting individuals is small, the system is discrete by definition, since the variety of opinion expressing different political points of view is limited.

Finally, the third class of systems is related to phenomena where the number of agents is so large to cover a continuous range of socio-economic states, but where the mathematical equations to describe the continuous phenomena are too complex to be solved from either an analytical or a computational point of view. In this case an approximation to a discrete decomposition is needed in order to develop a model that can be suitably analyzed and investigated. Moreover, a discrete approach is to be preferred when the investigation of the bridge from the microscopic scale to the macroscopic one is pursued, since it allows to capture the real states of the system without approximate it by means of mean variables.

The concept of functional subsystem is consistent with all these situations, subsequently in this present paper we choose to model the activity variable u as a discrete variable: in this case u belongs to the set

$$I_u = \{u_{-H} = -1, \dots, u_0 = 0, \dots, u_H = 1\}, \tag{1.3}$$

where the description of the state of each subsystem is delivered by the discrete distribution function

$$f_i^h = f_i^h(t): \quad [0,T] \times \mathbb{R} \to \mathbb{R}_+^n, \tag{1.4}$$

for $i = 1, \dots, n$ and $h = -H, \dots, H$.

As already said, the contents of this paper refer to the discrete case, while the interested reader can refer to [4] for a complete overview on functional subsystems in the continuous case.

We consider the general case of an open system in which the number n of subsystems can vary due to aggregation or fragmentation events and the number of possible socio-economic states is H. Taking into account the features listed in the previous section, we propose the following framework to describe microscopic interactions between subsystems, remembering that the definition of interactions and strategies is a key aspect of the modeling process since it affects the emerging behavior of the complex system.

• The *encounter rate* η_{ij}^{pq}, which depends on the interacting subsystems i and j and on the socio-economic and geometrical states, p and q respectively, of the interacting particles.

• The *transition probability density* $B_{ij}^{pq}(h)$, that describes the probability density that a candidate particle with state u_p of the subsystem i falls into the state u_h after the interaction with a field particle of the subsystem j with state u_q.

The transition probability density function is such that:

$$\forall i, j, \quad \forall p, q : \qquad \sum_{h=-H}^{H} B_{ij}^{pq}(h) = 1. \tag{1.5}$$

Remark 1. The encounter rate models the frequencies of interactions among functional subsystems. Although, according to these models there is no explicit graph structure among subsystems, the encounter rate defines an implicit stochastic network structure, describing which is the rate of interaction among different subsystems. In the more general case, the encounter rate varies according to both the geometrical variable and social activity.

Remark 2. The transition probability density, instead, defines the strategies and the rules of the stochastic games. The modeling of this term, which varies from application to application, is particularly delicate since it affects all the evolutionary dynamics of the process under consideration.

Let us now derive the evolution equation for the distribution function f_i^t corresponding to the above system which has to be regarded as a closed system:

The derivation of specific models refers to mathematical structures that have been proposed in [4] and [5] for closed and open systems. The structures described in what follows assume that active particles can interact within the same functional subsystem as well as with particles of the other subsystems, in general with different rates. Notice that the equation derived in this section take into account microscopic interactions among particles, without considering detailed multi-scale issues, which will be dealt with in the appendix. The mathematical structure, derived by conservation of active particles in the elementary volume of the state space, is as follows.

$$\frac{df_i^h}{dt} = \sum_{j=1}^{n} \left(\sum_{p=-H}^{H} \sum_{q=-H}^{H} \eta_{ij}^{pq} B_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=-H}^{H} \eta_{ij}^{pq} f_j^q \right) \tag{1.6}$$

The above equation can be properly modified when dealing with open systems, namely systems interacting with an outer environment, or when more complicated phenomena such as the passage of particles from one subsystem to the other are taken into account. Again, the interested reader may find the description of these aspects in the four papers present in the appendix.

Chapter 2

Summary of papers

A brief summary and critical analysis of the four peer reviewed papers [3], [4], [5], [6] are proposed in this chapter. The aim of this chapter is to provide the main ideas of each paper without giving excessive technicalities and mathematical details. The full length papers are reported in the Appendix.

2.1 New paradigms towards the modeling of complex systems in behavioral economics

The first paper [3], in press in the Elsevier journal *Mathematical and Computer Modelling*, provides a personal insight into complex social systems, focused on the assessment of specific features that play a key role in the mathematical interpretation, by models and, possibly, by a mathematical theory. Parts of the introduction to this thesis are taken from this paper, which provides a detailed overview on the subject dealt with.

This work provides a first conceptual approach towards the assessment of the paradigms which can act as a conceptual background for a unified approach to the modeling of a variety of social systems characterized by common features. It introduces the concept of stochastic games and network structure together with a multi-scale approach. These tools are also generated by the consideration that multi-scale aspects are an inner feature of all complex systems in general and of complex social systems in particular.

The paper summarizes and critically analyzes the main existing branches combining social sciences and mathematical modeling: agent-based models, game-theory, population dynamics, and social networks. A personal insight into complex living systems follows. Moreover it deals with the modeling of interactions between functional subsystems and active particles.

The basic idea consists in modeling economic and social phenomena by dealing with the specific system under consideration as a large population of active particles, playing stochastic games based on a dialogue that takes into account the particle activity distribution. Subsequently, the evolution equations on the probability distribution over the microscopic states of the interacting functional subsystems are derived. These equations act as paradigms to the modeling of specific systems and are derived taking into account the specific features regarding complex social systems that have been listed at the beginning of the paper.

The second part of this work aims at showing, by means of two specific applications, namely a nation subjected to the influence of internet media, such as blogs and online magazines and the job market, among firms and institutions set in a global context, how the general methodological approach presented in the first part can be technically applied. The examples are selected according to the personal author's experience and bias, while the aim consists in showing the treatment of the following issues:

• Selection of the activity variable and decomposition of the overall system into functional subsystems,

• Guidelines to model the stochastic interaction games and the strategic behavior of individuals.

The above analysis is developed at a qualitative stage first for closed systems and subsequently for open systems. Further aim consists in identifying the optimization problems generated by the application of external actions and their costs. Let us stress that this part of the paper has to be regarded as an introduction to issues that need a further development within a proper research program.

Finally, the last part focuses on the analysis of some multi-scale issues intrinsically related to the class of systems under consideration, developing a proper modeling procedure involving integral operators and/or gain and loss terms apt to capture the multi-scale features of complexity.

2.2 Towards a mathematical theory of complex socio-economical systems by functional subsystems representation

The second paper [4], published on the AIMS journal *Kinetic and Related Models*, shows how complex economic systems can be viewed as a network of several interacting functional subsystems, each of them characterized by the ability to express economic functions and purposes. The proposed mathematical representation defines a probability distribution over their functional states. Models in which functional subsystems express deterministically their functions, should be considered as particular cases of the more general stochastic description. The modeling aspects are developed taking into account some of the most relevant recent political-economic literature, among others [2], [8], [9], [29], [64], [80].

In details, this paper deals with a preliminary development of a mathematical theory to model complex socio-economic systems, where individual behaviors and interactions play a significant role on the evolution of the system itself. The mathematical approach takes advantage of the kinetic theory for active particles and of the concept of modules proposed by Hartwell [59], already cited in the above introduction of this thesis, for the interpretation of complex biological systems. Briefly, a module is viewed as an aggregate of entities specialized to develop a well defined function referred to the system under consideration. A modification of the objectives of the investigation may possibly modify also the structure of the modules. This paper develops the above concepts introducing the idea of functional subsystem that refers to both the functions expressed by the socio-economic system, which is the system under consideration, and the observation and representation scale used in the mathematical modeling process. The whole system is composed by several interacting subsystems, each related to a specific socio-economic function. The time-evolution of the system is described by differential equations delivered by a suitable development of the mathematical kinetic theory for active particles.

To give a simple and preliminary example, let us consider the case in which the overall system is a nation which, as visualized in Table 2.1, is decomposed into two or more subsystems identified by regional interest groups, which express, through specific actions taken by either their political parties or their peculiar interest groups, their attitude towards a process of secession by expressing a function u, which takes negative or positive values: when a certain region is expressing a positive value of u, then it is pro-secession, when it is expressing a negative value of u, then it is against it. Again, the absolute value of u measures the intensity of the expressed function.

Global nation			
Region1		Region2	
political parties	interest groups	political parties	interest group
$u > 0$ - pro	$u < 0$ - against	$u > 0$ - pro	$u < 0$ - against

2.1. Competition for a secession

The above example aims at showing how real socio-economic phenomena can be interpreted and decomposed using the concept of functional subsystems. The example technically shows a hierarchical decomposition from the global system to different subsystems, expressing the political activity u. This variable conceptually varies from one case to the other: this means clearly that the specific meaning of u is strictly related to the phenomenon under consideration. Even the structure of the decomposition of the global system varies corresponding to the different examples shown above; this depends on which different aspects of the phenomenon it is interesting to focus on.

Bearing this example in mind, let us give some mathematical details regarding some preliminary modeling aspects. Let us consider a network of n interacting functional subsystems whose function is identified by the variable $u \in \mathbb{R}$, where the value $u = 0$ separates the positive and negative valued *functions* expressed by each subsystem. The overall state of the system is described by the probability distributions:

$$f_i = f_i(t,u) : \quad [0,T] \times \mathbb{R} \to \mathbb{R}^+ \quad f_i \in L_1(\mathbb{R}), \tag{2.1}$$

where the subscript refers to the ith subsystem, and where for each of them:

$$\lim_{|u|\to\infty} f_i = 0, \qquad \int_{\mathbb{R}} f_i(t,u)\, du = 1, \quad \forall\, t \geq 0, \tag{2.2}$$

Each f_i has the structure of a probability density such that

$$\int_a^b f_i(t,u)\, du = P(t; u \in [a,b]) \tag{2.3}$$

is the probability that the ith subsystem expresses a function in the range $[a,b]$.

In the first part of the paper this mathematical framework is enriched, also taking into account open systems, namely systems able to interact with entities belonging of some outer environment.

In the following of the paper the concepts hereby exposed are developed with this scheme:

i) Characterization of the model of competition for a secession;

ii) Sample simulations focused to visualize the role of the parameters of the model on the behavior of the solutions;

iii) A critical analysis to develop and improve the model.

Once the model has been characterized with the introduction of parameters and the derivation of a set of differential equations describing the evolution of the system, the goal of the simulations is to investigate the role of some key parameters, given random uniformly distributed initial conditions. Some interesting results have been shown, that highlight how the model is able to reproduce some practical aspects of currently observed phenomena.

2.3 On the modelling and simulation of the competition for a secession under media influence by active particles methods and functional subsystems decomposition

The third paper [5], published on the Elsevier journal *Computers and Mathematics with Applications* is devoted to the application of the above mentioned mathematical methods to a specific socio-economic system, chosen among the ones described in Paper 2, taking into account the fundamental role of external actions. The proposed mathematical structure dealt with is the same as in [4], focusing on the discrete case.

The recent socio-economic literature has shown an increasing interest in the development of models regarding complex political phenomena, as already seen in [8], [29], [7], even if from a theoretical different approach, which avoid the concept of complex social dynamics. Some interesting references can be found in [54], [53] and [55], where socio-political phenomena such as dictatorships, terrorism and strategic interactions among interests groups are modeled and discussed from a quantitative approach, which takes into account dynamical interactions among agents.

Moreover, while the subject of media influence has been deeply treated in the modern socio-political literature, as witnessed by, for example, [61], [65], [51],[19], the present literature shows few examples of the discussion of socio-economic phenomena under the influence of media, from a quantitative approach.

The system under consideration is a nation which is decomposed into two or more subsystems identified by regional interest groups, which express, by means of specific actions taken by either their political parties or their peculiar interest groups, their attitude towards a process of secession by expressing a function u, which takes negative or positive values: when a certain subsystem is expressing a positive value of u, then it is pro-secession, when it is expressing a negative value of u, then it is against it. The absolute value of u measures the intensity of the function expressed. Another element which plays an important role in the system is the so-called external action, played by media: in our system, each interest group or party can be strongly affected by media, which can be viewed as an additional external functional subsystem; this means that media is another very powerful element of the decomposition. Given this decomposition, the paper focuses on a coherent mathematical modeling of the external action exerted by media, defining strategic behaviors that media may assume: once a specific purpose is designed, media have to choose a strategy to reach this purpose at their best. In addition, media influence just lasts for a certain window of time, given a certain level of investment, and, after this period, expires, so that interactions are the same as without it. Clearly, the bigger is the investment, the longer is the influence. The modeling procedure leads to a system of differential equation that rules the evolution in time of the considered phenomenon.

Subsequently, an analysis finalized to the selection of the optimal strategy, based on numerical simulations which refer to each strategy, is developed. These simulations are obtained by solving the initial value problems generated by the already mention system of differential equation linked to suitable initial conditions.

A comparison among the various simulations is finally proposed with the aim of focusing on emerging events, due to the action of media. It is remarkably evident that all the defined strategies significantly influence opinion dynamics; moreover, all the three strategies are different one from the other in their results, even reaching the common aim.

2.4 Fashion: an agent based-model

The forth paper [6], deals with the derivation of a suitable agent based model, regarding the phenomenon of fashion. This paper shows different mathematical tools with respect of the other three: an agent based models and its dynamics involving two categories of agents are derived.

Fashion has been studied in different fields of social sciences, with different aims and perspectives. It is a highly interdisciplinary subject of investigation and it has risen the attention of scholars belonging to many different fields: from sociology to anthropology, psychology and economics. Among others [18], [27], [1], [39], [30].

A variety of models are available in the literature: from classic micro-economic minimizing and maximizing problems, to game theoretical approaches. This paper aims at contributing to the existing literature by investigating the behavior of two classes of agents, each of them set in the same environment and able to reciprocally interact.

A population composed by $N+K$ agents is first considered, each of them described by a uniformly distributed state, a parameter that can be considered related to the concept of *taste*. Within this population, two classes of agents are taken into account: the common agents $i = 1,...,N$, who decide to consume as a consequence of everyday life interactions; and the guru agents $k = 1,...,K$, who are assumed to be a trend setter, having an active role in influencing the common agent consumption desire. The number K of guru agents are assumed to be smaller than the number N of common agents, namely $K << N$. Each agent is characterized by a utility function representing its degree of satisfaction at every time t. This utility function depends on the consumption choices made by agents at t. Common agents and gurus have different utility functions, depending on various parameters.

Once the classes of agents are introduced, the dynamics are defined. Common agents and gurus have different types of dynamic behavior. Different kinds of behavior are described both for common agents and for gurus. Thanks to numerical simulations, the different kinds of behavior are analyzed and compared, with regard to the variation of the key parameters of the model.

The goal of this paper is to provide hints towards the investigation of an intriguing phenomenon of our times that has very important implications in a wide variety of socio-economic disciplines.

Chapter 3

Research perspectives

This dissertation focused on the mathematical modeling of complex socio-economic phenomena, where individual human behaviors play a crucial role in determining the emerging collective dynamics.

Several types of socio-economic systems are such that the collective dynamics are determined by complex individual interactions, so that the sum of the individual behaviors does not straightforwardly lead to the description of the collective dynamics. Stochastic games define the output of the interactions when the input states are given. Personal behaviors generate not only deviations from the most likely inputs, but may possibly change due to environmental conditions, for instance the onset of panic situations.

The interest in complex socio-economic systems has witnessed a remarkable increase in recent years, thanks to the growing awareness of the fact that many systems related to human behaviors cannot be successfully modeled by traditional methods: new investigations leading to new methodologies are called for. This dissertation aims at providing a preliminary contribution in this direction, keeping in mind that further research is required in order to develop a robust mathematical theory.

Much remains to be done. We now provide our personal view of the most urgent further developments of the work reported in this dissertation.

The modeling approach proposed in this dissertation is based on the concept of functional subsystem, considered as an aggregation of groups of interest expressing a common socio-economic function. Therefore, if the general context of the mathematical model changes, the characterization and the size of the functional subsystems may also need to change. In principle, the function expressed by a subsystem can be described as a vector with several components; however, the complexity inherent in modeling interactions suggests to refine the identification of the subsystem by an additional decomposition, in order to obtain that each subsystem expresses one scalar function only. Our modeling approach is based on the assumption that the number of functional subsystems is constant in time. However, the aggregation or fragmentation of agents into subsystems may possibly occur in many circumstances during the system dynamics. For instance, aggregation or fragmentation may occur as a consequence of the size of the subsystems, or the presence of external processes. Including these possibilities in our mathematical models can add a new dimension to the global system dynamics. However, it must be observed that this goal cannot be pursued by a straightforward generalization of the approach proposed in this dissertation, since a further development of the mathematical structure cannot describe aggregation or fragmentation events, unless properly modified.

The proposed modeling approach accounts for interactions that do not depend on the geometry of the system, and specifically on the geographic position of the interacting subsystems. This simplification can be considered adequate when communications among agents occur through delocalized devices: for instance the media, the telephone, the Internet, and similar. On the contrary, communications among agents may be constrained by networks that organize and select the dialogue between and among functional subsystems. Like in the previous case, accounting for locality of interactions requires a further development of the mathematical modeling framework, to incorporate network structures. Accounting for this aspect in the mathematical modeling framework may allows the investigation of interesting new aspects, like the influence of the network structure and its degree of clustering on the overall system dynamics.

Multi-scale issues, that were briefly analyzed in the third paper, should be further investigated, and included in the modeling of some application. The problem of the interactions between the microscopic and the macroscopic scales deserves further investigations, leading also in this case to the development of new mathematical techniques. A crucial improvement in the investigation could consist in the the deduction of the macroscopic behaviors of the system by means of the limit solutions of the differential equations derived on the microscopic state space.

As far as agent based models are concerned, let us remember that this methodology can be considered a complement to KTAP, in the sense that the conclusions derived from an agent-based model can provide hints for further developments of KTAP, adapted to specific classes of phenomena. Therefore, both for KTAP and agent-based models, specially valuable would be a proper calibration of parameters, according to data collected from the observation of instances of specific phenomena under consideration. This step would provide an essential contribution regarding the verification of the applicability of the models.

Further details regarding possibilities for the development and improvement of the mathematical models discussed in this dissertation are included in the conclusive sections of each paper in the Appendix.

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Chapter 4

Appendix

New Paradigms Towards The Modelling of Complex Systems in Behavioral Economics

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Abstract

This paper develops a mathematical framework based on the kinetic theory for active particles and on a suitable decomposition into functional subsystems and shows how it can be implemented to describe some specific complex economic applications. Specifically, the applications are focused on opinion dynamics and job mobility phenomena. These two examples offer a first insight into multi-scale issues: starting from the application, a preliminary mathematical framework taking into account both microscopic and macroscopic interactions is developed. This framework may be adapted to the modelling of a great variety of complex phenomena.

Key words: **Keywords**— Kinetic theory, stochastic games, active particles, socio-economic systems, evolution and dynamics.

1. Plan of the Paper

The development of a mathematical description of complex socio-economic systems is a fascinating, however difficult, perspective of interaction between mathematical and social sciences. An important objective consists in the assessment of the paradigms which can act as a conceptual background for a unified approach to the modelling of a variety of social systems characterized by common features.

This program is pursued by suitable developments of the so called mathematical kinetic theory for active particles KTAP [1], [2], which has shown to be a useful reference applications in various fields of life sciences such as the immune competition [3], traffic flow [4], social dynamics [5], [6], psychological interactions [7], [8].

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A modelling approach to complex socio-economic systems has been recently developed using the mathematical tools offered by the KTAP theory [9], [10]. These papers offer various hints to revisit this theory by introducing the concept of stochastic games and network structure together with the multiscale approach already introduced in [11]. In particular, multiscale aspects are an inner feature of all complex systems in general and of complex social systems in particular. It is worth remarking that the common features of complex systems allow to update mathematical tools in different fields of life sciences and transferring them, after suitable developments, from one field to the other.

Several attempts to combine social sciences and mathematical modelling can be found in the literature: in particular the main branches are agent-based models, game-theory, population dynamics, and social networks.

Agent-based models are computational models in which interactions among individuals (agents) are simulated, in order to understand the emerging behavior of the macroscopic overall system, starting from the microscopic dynamics of each agent. The goal is to find the equilibria or the emergent patterns in the complex system, as a consequence of the dynamics of the agents. This approach combines elements of game theory, multi agent systems and Monte Carlo methods. These models consist in systems of equations that simulate the simultaneous behavior of multiple agents, starting from specific sociological assumptions, in order to recreate and predict the global behavior of the system.

Agent based models can be considered a complement to traditional analytic methods, since they may recreate the equilibria or non-equilibria resulting from the analytic method itself. Indeed, they can explain the emergence of unpredictable patterns and distinguish among different types of network structures. These models have been recently used to describe a great variety of network-related phenomena of recent times, like internet, terrorism, traffic jams, financial crisis, consumer behavior, spread of epidemics and social segregation. Notice that all the phenomena listed above are strongly affected by individual or particles interaction and behavior. Some key references can be found in [12] and [13].

Game theory is a branch of applied mathematics that has become widely used in social sciences and economics. The goal of game theory is to describe behaviors of individuals (players), who have to design specific strategies to give the best response to other players' choices. Game theory is widely applied in all the rational modelling of social sciences. It aims at finding equilibria in the sets of strategies and different kinds of equilibria have been defined. Starting from the analysis of economic competition with the the 1944 book *Theory of Games and Economic Behavior* by John von Neumann and Oskar Morgenstern [14], game theory it is now widely used to model biological, social and even telecommunication interactions. Most notable references can be found in [15], [16] and [17].

Population dynamics, instead, studies how the number of individuals in one or several populations changes, under the action of biological and environmental processes. It focuses on the rise and decline of different populations of different particles: individuals, cells and so on. It has been a dominant branch of mathematical applications to biology for more that 200 years. In the early 19th century it widely investigated demographic studies, whose main modelling aspects are related to the following dynamic rate functions of the considered populations: natality or birth rate, growth rate and mortality. For a complete survey on the subject it is possible to refer to [18].

Particularly important is, in economic and social applications, the use of population dynamics with internal structure, where an additional internal variable can model specific characteristics of the system under consideration. Large systems of interacting individuals can be modelled by systems of partial differential equations taking into account internal variables describing the socio-economic state of the individuals. This internal structure describes the peculiarities of the system under consideration, and contributes to model the emergence of a collective behavior. An introduction to population dynamics with internal structure is given in [19],[20] and [21].

Notice that, in population dynamics, the internal structure is a deterministic variable, in the KTAP, instead, the internal variable is heterogeneously distributed among agents, while interactions modify such distribution. Indeed, this is an essential ingredient to model social systems in the framework of behavioral economics.

The study of *social networks* is receiving growing attention, focused on the role of networks in determining and constraining social behavior. Starting from the Milgram's experiment [22], and the celebrated paper developing the concept of *small-world phenomena* [23], it is possible to find, in the recent literature, many socio-economic phenomena studied by this approach [24], [25], [26]. In particular, [27] and [28] show the importance of social networks and interactions in economics. The wholesale fish market in Marseille has been studied in [27], introducing the idea of high loyalty of buyers to sellers, and persistent price dispersion taking place in the same place every day. The authors build an adaptive agent model, where sellers set quantities, prices and treatment to customers, buyers instead decide which seller to visit and at which price to buy. Subsequently, the emerging behavior is analyzed. The second work [28], instead, is a collection of articles having in common the mathematical modelling approach to economics, which takes into account *heterogeneous interacting agents*. In its introduction, the reconciliation among economics, psychology and sociology, is highlighted thanks to the new cognitive approach towards economics in general.

The borders among these disciplines show, in practical applications, various flexibilities, in the sense that tools and methods of each of them are used together for a better development of mathematical models. In particular, a recent field of investigation, which wisely combines elements of the three branches described above is the so called *behavioral economics*. Its principal scope is to merge properly psychology and economics, stressing the importance of human behavior and psychological attitude in the evolution of complex systems of individuals. Behavioral economics widely use experiments as a preliminary step of the modelling process. To obtain an idea of the wide applications and philosophical foundations of behavioral economics it is possible to refer to [29]-[39]. Moreover [40] and [41] provide an insight into the links between behavioral economics and game theory, in order to adapt the second to the challenges proposed by the first, introducing the concept of *behavioral game theory*. Additional references can be found in [42] and [43].

It is plain that stochastic aspects should characterize models derived within the framework of the behavioral economy, consequently models derived by KTAP methods, seem to be appropriate to investigate this kind of phenomena. This theory can be briefly summarized as follows. Social systems are characterized by a large number of particles representing the interacting socio-economic entities, called active particles, (consumers, firms, institutions and so on), whose state may be characterized by both mechanical variables and an activity variable representing the specific socio-economic feature of the system itself. While the activity variable describes the microscopic state of the system, the global state is described by the probability distribution function over the microscopic state whose weighted moments recover the macroscopic quantities of the system. Notice that thanks to the activity variable, the output of the interactions is a direct consequence of the strategic behavior which takes place among active particles. The activity variable, in the more general case, can be different from particle to particle. The evolution in time and space of the system is derived by a balance of the inflow and outflow of active particles in the elementary volume of the space, as a consequence of the dynamics based on strategic interactions.

Consequently, the state of the interacting active particles is described by random variables, while the output of interactions is again delivered by random features.

Notice that this theory borrows and extends mathematical tools from the four branches described above. As in agent-based models, models derived from the KTAP may be characterized by multi-agent (or multi-particle) interactions defined as a consequence of assumptions regarding the sociological features of the system under consideration, whereas the mathematical tools used are more refined since they generally allow to obtain systems of partial differential equations to be solved with computational methods. With regard to game theory, KTAP models describe particle interactions thanks to the definition of stochastic games, which characterize the particle response after the encounter with other particles. As in game theoretical models, the evolution of the system is affected by the definition of these games, even if the overall mathematical framework leads to systems of equations that are not present in traditional game theoretical models. The KTAP distinguishes from population dynamics since it takes into account an internal variable stochastically defined among particles, which better represents the random feature of human behaviors: in the KTAP models, indeed, the socio-economic state of the system is described by a probability density function. Finally, since the KTAP models describe the evolution of systems as a consequence of particle interaction, a network structure may arise in the mathematical framework of the model, helping the description of the interactions within the system. Concepts belonging to all these four mathematical methods are, then, taken into account and updated to develop a new mathematical approach, based on the concept of functional subsystem, see [9], [10]. These papers extend the preliminary works, [5], [6], [44] where, for the first time, social applications studied with the tools offered by the KTAP are investigated.

The following sections show and analyze in details this new approach. The contents are presented in five more sections that follow the above introduction. In details:

- Section 2 proposes a personal insight into complex living systems, of which socio-economic systems are a subset, focused on the assessment of specific features that play a key role in the mathematical interpretation by means of models and, possibly, of a mathematical theory. This section provides various concepts that act as guidelines for the formal treatment developed in the following sections.

- Section 3 deals with the decomposition of the overall systems into functional subsystems according to the definitions proposed in [9]. As we shall see, the decomposition is a method to reduce the complexity of the overall system, while the representation uses random variables. This section also introduces a suitable nomenclature that is used in the sections that follow. Moreover it deals with the modelling of interactions between functional subsystems and active particles. The basic idea consists in modelling the system as a large system of active particles that play stochastic games among them, based on a dialogue that takes into account their activity distribution. Subsequently, the evolution equations on the probability distribution over the microscopic state of the interacting functional subsystems are derived. These equations act as paradigms to the modelling of specific systems.

- Section 4 shows how several applications can be developed by using the mathematical structures derived in the preceding section. Subsequently, some further developments are outlined taking advantage of the analysis of the afore mentioned applications.

- Section 5 proposes a critical analysis of the contents of the preceding sections mainly focused on research perspectives. The main issue which is dealt with is the analysis of some multiscale problems intrinsically related to the class of systems under consideration.

2. Basic Aspects of Complex Economic Systems

Let us consider a large system of interacting particles that belong to physical systems whose dynamics are determined by their ability to dialogue among themselves and develop specific strategies. Typical examples are consumers in the market, cells of multicellular systems in biology, pedestrians in crowds, vehicles-driver on roads, or animals in swarms.

If the strategy changes according to the number of interacting particles, all these systems can be classified as complex systems since the dynamics of a few entities does not generate straightforwardly the dynamics of the whole system. Consequently, emerging collective behaviors do not appear to be straightforwardly related to the individual dynamics.

The various systems which have been listed above belong to the living matter. Therefore, their dynamics should include the description of essential aspects of living systems such as ability to develop specific strategies, competition, evolution, as well as reproductive and/or destructive ability. Although these systems belong to different fields of life sciences some common features can be identified and, ultimately, it is reasonable querying if it is possible extracting some common features for all of them.

This section deals with a personal assessment of some common and distinctive features characterizing the class of systems under consideration in view of transferring such a phenomenological analysis into a suitable mathematical framework, which claims to act as a paradigm for the modelling of complex social systems. For a detailed overview of complexity and socio-economic sciences, see [45].

• ***Complexity and decomposition into functional subsystems***

Complexity is the most challenging feature of socio-economic systems: a huge number of stochastic interactions may generate emerging patterns of strategic behaviors, which vary according to the number and the socio-economic state of the interacting particles. As a consequence, the problem of a suitable decomposition, apt to reduce the complexity of the system arises, in order to properly classify and model interactions. This is the reason why the concept of functional subsystem is introduced: more in details, the global framework, which corresponds to a specific socio-economic phenomenon, can be hierarchically decomposed into *functional subsystems*.

Each functional subsystem expresses a specific social function, called *activity* which plays an active role in the evolution of the global phenomenon and which can be properly mathematically modelled. Notice that the activity can be regarded as a continuous or discrete variable, depending on the applications. Particles may refer, according to the selected scale, to individuals, institutions, firms, interests groups, companies or every constituent element of a socio-economic society. Those particles expressing the same activity function are then grouped into the same functional subsystem: so that the overall system is decomposed in many functional subsystems, each of them expressing different abilities and purposes. Notice that this decomposition reduces the number of interacting entities and, as a consequence, the complexity of the system. Moreover, this procedure may help the modelling aspects, since it allows to treat and model separately different groups of particles with different behaviors and different strategies.

The dynamics is then defined through interactions among functional subsystems. Notice that interactions undergo double dynamics: inner dynamics due to interactions among particles within the same functional subsystem and outer dynamics due to interactions among different functional subsystems. This decomposition is a flexible approach that can be adapted to many particular investigations and varies according to the phenomenon under consideration.

• ***Interactions.***

An additional key feature of complex systems is the network of interactions among the different components of the system itself: since interactions are determinant in the emergence of a collective strategic behavior, namely in the modification of the state or of the strategy of the interacting particles, the definitions of the type and structure of interactions become essential, in the modelling procedure. Notice that the mathematical modelling of interactions cannot disregard the concept of functional subsystem: it has to be clear which subsystems are interacting and under which conditions.

• ***Distance and topology.***

Another problem which arises in the mathematical modelling of living systems refers to how their topological and state distribution affects interactions. Subsystems and particles may interact when they are sufficiently close either in a geometrical sense or in a microscopic state sense. In the first case interactions take place according to the spatial localization of particles and subsystems, in the second case according to their socio-economic state distribution. As a consequence, it seems appropriate to introduce the concepts of geometrical and state distances: keeping in mind that each particle or subsystem expresses a specific activity, which represents its social state, and it is physically set in the geometrical space, it is possible to define a distance in the expressed social function and a distance in the spatial localization. Subsystems with a low state distance are supposed to be close in their tasks and purposes and consequently to interact more often than subsystems with a high state distance.

On the other hand, subsystems with a low geometrical distance may interact more easily than subsystems characterized by a far space localization. Therefore, the frequency and the quality of interactions varies according to a balance of this two distances. Notice that the rules with which subsystems and particles interact, may define a network structure which depends both on the topological and on the state distribution of the system, whose links represent the possible existing connections among functional subsystems and particles. Links can be deterministically or stochastically modelled, depending on the quality of interactions. Once a network topology is modelled, it becomes possible to define other concepts of distances related to the number of hops which separate subsystems or particles on the network itself.

• ***Stochastic Games.***

According to their purposes, functional subsystems play specific strategic games, whose outcomes depend on the topological and state distribution of the interacting subsystems. Since, generally, the strategy of particles belonging to the same functional subsystem is randomly distributed, the final outcome is not deterministically predictable, therefore it must be modelled, in the more general case, from a stochastic point of view; this is the reason why the game is said to be stochastic. Notice that strategies are not constant on time: each subsystem may change strategies according to its geometrical position and its distribution of state, within the system, at every time t. Moreover, while the modelling of simple phenomena takes into account binary interactions, for a great variety of models it is necessary to study interactions stochastically distributed among every functional subsystem, increasing the complexity of the system.

• ***Multiscale aspects.***

Multiscale aspects are probably one of the main and most fascinating features of complexity, since they have not been properly investigated yet in economics and social sciences. First of all, the proper scales have to be defined regarding the different phenomena under consideration: the suitable scales to model individual interactions are not the suitable ones to model state building processes or companies and institutions network structures: as a consequence, the definition of the micro and the macro level becomes a key modelling aspect. Moreover, in many living systems, the dynamics at the microscopic scale of the active particles affect the dynamics at the higher scale of the functional subsystems and vice versa: decisions taken by individuals (the active particles) working in a firm (the functional subsystem) may influence the strategic behavior of the firm with institutions and other companies (other functional subsystems), while the competition among firms may influence the strategic behavior of individuals within the same firm, just to give a very generic example.

3. Modelling Interactions among Functional Subsystems and Stochastic Games

Socio-economic systems are generally characterized by a large number of interacting entities, all of them expressing different functions and strategies. Therefore, it is necessary developing methods able to reduce the overall complexity.

The modular approach proposed by Hartwell [46] and, to some extent by [47] can be developed by introducing the concept of socio-economic functional subsystem according to the following definition:

a functional subsystem is an assembly of socio-economic agents with the ability to express collective specific strategies that play a fundamental role in the dynamics. More specifically, depending on the phenomenon under consideration, the socio-economic framework is decomposed into sub-groups, called functional subsystems, having the same socio-economic aim and expressing the same activity variable.

Notice that even the decomposition into functional subsystems is strictly related to the modelling investigation process: depending on which features of the problem it is interesting to focus on, the functional subsystems are properly constructed. This framework analyzes the evolution of the socio-economic activity variable, as a result of interactions among the different subsystems of the global structure.

One of the advantages of this method is that it creates a bridge among the microscopic and the macroscopic scale. The concept of functional subsystem lyes in a intermediate scale which provides the link between the microscopic level (which describes the physical state of each particle) and the macroscopic one (which describes the physical state of the overall system).

Hereby a mathematical framework regarding the concept of functional subsystem is provided. The overall state of each functional subsystem is described by the distribution function and it is viewed, according to the analysis of Section 2, as a network of interacting functional subsystems. Their representation is based on the assumption that each of them has the ability to express a specific activity, identified by the variable $u \in \mathbb{R}$, where the value $u = 0$ separates the positive and negative valued ***functions*** expressed by each subsystem:

$$f_i = f_i(t, \mathbf{x}, u) : \qquad [0,T] \times D_{\mathbf{x}} \times D_u \to \mathbb{R}_+, \tag{3.1}$$

where the subscript $i = 1, ..., n$ identifies the ith functional subsystem, $\mathbf{x}$ the geometrical microscopic state, u the activity variable. f_i is called generalized distribution function and $f_i(t, \mathbf{x}, u) d\mathbf{x} du$ represents the number of active particles which at time t are in the elementary volume of microscopic states $D_{\mathbf{x}} \times D_u$. Notice that this representation corresponds to the fact that social systems are identified by the geometrical variable and the social state of their components, namely the microscopic state is not scalar but defined by the vector $(\mathbf{x}, u)$. These two variables are those that play a role in the stochastic strategy dynamics described in the previous section.

The assessment of the role of the space variable needs further reasoning. Indeed, this variable in several interesting systems of social and economic sciences does not play an important role, considering that the communication among active particles is not influenced by localization, while, in other cases, the space variable identifies the region where a functional subsystem is localized. In this latter case, $\mathbf{x}$ identifies the functional subsystem by its localization to be also related to its activity. Therefore, the representation is given by

$$f_i = f_i(t,u) : [0,T] \times D_u \to \mathbb{R}_+, i = 1,...,n \tag{3.2}$$

The analysis proposed in what follows refers to the representation (3.2).

The activity variable u can be a discrete or a continuous variable depending on the phenomena under consideration and on the constraints of the modelling process. More in details, the socio-economic function described by means of the activity variable can contribute to the modelling of different kinds of discrete systems at the microscopic scale. In particular, continuous systems which are discretized in order to be measured, discrete systems due to the fact that the number of interacting agents is limited and continuous systems that are modelled by means of a discrete filter.

Regarding the first type of systems a well-known example is the electoral competition of a large country, where the socio-political state can assume every single value corresponding to the personal political opinions of individuals; during the voting process, however, opinions are forced to be discrete since every agent has to express a preference on a limited number of parties.

As far as the second class of systems is concerned, if we think, for instance, of a voting process in a local assembly, where the number of voting individuals is small, the system is discrete by definition, since the variety of opinion expressing different political points of view is limited.

Finally, the third class of systems is related to phenomena where the number of agents is so large to cover a continuous range of socio-economic states, but where the mathematical equations to describe the continuous phenomena are too complex to be solved from either an analytical or a computational point of view. In this case an approximation to a discrete decomposition is needed in order to develop a model that can be suitably analyzed and investigated. Moreover, a discrete approach is to be preferred when the investigation of the bridge from the microscopic scale to the macroscopic one is pursued, since it allows to capture the real states of the system without approximate it by means of mean variables.

The concept of functional subsystem is consistent with all these situations, subsequently in this present paper we choose to model the activity variable u as a discrete variable: in this case u belongs to the set

$$I_u = \{u_{-H} = -1, \ldots, u_0 = 0, \ldots, u_H = 1\}, \tag{3.3}$$

where the description of the state of each subsystem is delivered by the discrete distribution function

$$f_i^h = f_i^h(t) : \quad [0,T] \times \mathbb{R} \to \mathbb{R}_+^n, \tag{3.4}$$

for $i = 1, \ldots, n$ and $h = -H, \ldots, H$.

As already said, the contents of this paper refer to the discrete case, while the interested reader can refer to [9] for a complete overview on functional subsystems in the continuous case.

We consider the general case of an open system in which the number of particles within subsystems can vary due to switching events events and the number of possible socio-economic states is H. Taking into account the features listed in the previous section, we propose the following framework to describe microscopic interactions between subsystems, remembering that the definition of interactions and strategies is a key aspect of the modelling process since it affects the emerging behavior of the complex system.

• The ***encounter rate*** η_{ij}^{pq}, which depends on the interacting subsystems i and j and on the socio-economic and geometric states, p and q respectively, of the interacting particles.

• The ***transition probability density*** $B_{ij}^{pq}(h)$, that describes the probability density that a candidate particle with state u_p of the subsystem i falls into the state u_h after the interaction with a field particle of the subsystem j with state u_q.

The transition probability density function is such that:

$$\forall i,j, \quad \forall p,q: \qquad \sum_{h=-H}^{H} B_{ij}^{pq}(h) = 1. \tag{3.5}$$

Remark 3.1 *The encounter rate models the frequencies of interactions among functional subsystems. Although, according to this model there is no explicit graph structure among subsystems, the encounter rate defines an implicit stochastic network structure, describing which is the rate of interaction among different subsystems. In the more general case, the encounter rate varies according to both the geometric variable and social activity.*

Remark 3.2 *The transition probability density, instead, defines the strategies and the rules of the stochastic games. The modelling of this term, which varies from application to application, is particularly delicate since it affects all the evolutionary dynamics of the process under consideration.*

Let us now derive the evolution equation for the distribution function f_i^h corresponding to the above system which has to be regarded as a closed system:

The derivation of specific models refers to mathematical structures that have been proposed in [9] and [10] for closed and open systems. The structures described in what follows assume that active particles can interact within the same functional subsystem as well as with particles of the other subsystems, in general with different rates. Notice that the equations derived in this section take into account microscopic interactions among particles, without considering detailed multiscale issues, which will be dealt with in the last section. Let us first consider closed systems. The mathematical structure, derived by conservation of active particles in the elementary volume of the state space, is as follows.

$$\frac{df_i^h}{dt} = \sum_{j=1}^{n} \left(\sum_{p=-H}^{H} \sum_{q=-H}^{H} \eta_{ij}^{pq} B_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=-H}^{H} \eta_{ij}^{pq} f_j^q \right) \tag{3.6}$$

Let us now consider open systems, that interact also with the outer environment. We have to deal with functional subsystems interacting with external agents identified with the subscript $\ell = 1,...,m$. The external subsystem ℓ has the ability to influence the socio-economic state u of the subsystem i, through a particular action identified by the variable v. The analysis developed in what follows is based on the assumption that the action of the outer agents can be modelled as:

$$n_{i\ell}(t)\, g_\ell(v)\,, \tag{3.7}$$

where $n_{i\ell} = n_{i\ell}(t)$ is the intensity, that depends on time, during which the agent ℓ acts on the subsystem i; and $g_\ell(v)$ is the probability density associated to the variable v that characterizes the action of the outer system, namely the probability that this action takes place over system i.

Notice that each external agent is regarded as a specific functional subsystem with the ability to interact with the functional subsystems of the inner system. Both terms $n_{i\ell}$ and g_ℓ are supposed to be given functions of their arguments.

The mathematical framework to describe the microscopic interactions between a subsystem i and an external agent ℓ needs the modelling of the: ***Inner-outer transition probability density*** $G_{i\ell}^{pq}(h)$, that describes the probability density that a candidate particle with state u_p of the subsystem i falls into the state u_h after the interaction with a field particle of the external agent ℓ with state v_q.

The interaction term $G_{i\ell}^{pq}(v_q, u_p; u_h)$ satisfies the following condition:

$$\forall i, \ell, \qquad \forall p, q: \quad \sum_{h=-H}^{H} G_{i\ell}^{pq}(h) = 1. \tag{3.8}$$

Remark 3.3 *The reasonings of Remark 3.2 apply also to the inner-outer transition probability density: also this operator contributes to design an implicit interaction graph structure and to set strategies and rules of the stochastic games among functional subsystems and external agents.*

Let us now derive the evolution equation for the distribution function f_i^h corresponding to the above open system:

$$\begin{aligned}\frac{df_i^h}{dt} &= \sum_{j=1}^{n} \left(\sum_{p=-H}^{H} \sum_{q=-H}^{H} \eta_{ij}^{pq} B_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=-H}^{H} \eta_{ij}^{pq} f_j^q \right) \\ &+ \sum_{\ell=1}^{m} \left(\sum_{p=-H}^{H} \sum_{q=-H}^{H} n_{i\ell} G_{i\ell}^{pq}(h) g_\ell^q f_i^p - f_i^h \sum_{q=-H}^{H} n_{i\ell} g_\ell^q \right).\end{aligned} \tag{3.9}$$

The above interaction terms do not take into account the variation in time of the number of particles within functional subsystems: additional events can be taken into account considering that they may play a relevant role in the time evolution of social and economic systems. In particular, the case of active particles which may move from one functional subsystem to the other can be considered, as a consequence of interactions among subsystems.

This phenomenon happens often in socio-economic systems. For instance, let us consider the classic example of elections: we may think to decompose the electoral systems into functional subsystems related to political parties and interest groups, dynamically interacting in the political scenario. It may happen that individuals (corresponding to particles in the mathematical framework) belonging to a specific party or interest group change advice and switch to other political institutions, modifying the global dynamics of the system.

The above phenomenon can be described by the: ***Switching rate*** $\Gamma_{rs}^{pq}(i,h) \in \mathbb{R}$ which depends on the interacting subsystems r and s and on the socio-economic states, p and q, of the interacting particles and indicates the rate with which a particle of subsystem r with state p switches to subsystem i with state h after the interaction with a particle of state q belonging to subsystem s; and by the ***Inner-outer switching rate*** $\Lambda_{r\ell}^{pq}(i,h) \in \mathbb{R}$ which depends on the inner and outer interacting subsystems, respectively, r and ℓ and on the socio-economic states, p and q, of the interacting particles and indicates the rate with which a particle of subsystem r with state p switches to subsystem i with state h after the interaction with a particle of state q belonging to the outer subsystem ℓ.

Both these terms satisfy the following properties:

$$\forall r,s,p,q, \qquad \sum_{i=1}^{n}\sum_{h=-H}^{H}\Gamma_{rs}^{pq}(i,h)=1, \qquad \forall r,\ell,p,q, \qquad \sum_{i=1}^{n}\sum_{h=-H}^{H}\Lambda_{r\ell}^{pq}(i,h)=1, \tag{3.10}$$

Remark 3.4 *These additional terms contribute to develop and define the mathematical modelling of stochastic games and strategies. In addition, the switching rates can define multi level dynamics: these rates set the dynamics at the functional subsystem level starting from the dynamics at the particle level and vice versa. The modelling of the switching rates must take into account both the state of the interacting particles and the subsystem which they belong to: the resulting dynamics cannot disregard this two level scale interaction. In details, the definition of this rate may involve the interactions among functional subsystems at the higher scale and the interactions among particles both within the same subsystem and among different inner and outer subsystems of the global environment. These interactions affect the global functional subsystem architecture, in the sense that they change the number and the state of their microscopic components.*

Another phenomenon that we do not take into account in this paper, but that can be further investigated is the one related to the fact that the number of particles within the overall subsystem may evolve in time due to inlet and/or outlet events. To stress this matter, let us consider again the electoral competition system. In this case individuals belonging to a specific subsystem may decide to abandon the political scenario as a consequence of interactions among subsystems or new individuals may enter the competition. This means that the total number of particles is not constant but is changing over time. Notice that as a consequence of the variation of the number of particles, new functional subsystems may appear and old ones may disappear.

The modelling of the interactions terms allows the derivation of a suitable evolution equations for the distribution functions f_i^h corresponding to the $i-$th functional subsystem and to the h-state. The derivation is obtained equating the increase in time of active particles in the elementary volume of the state of the microscopic states. Technical calculations based on conservation of particles in the space of elementary states yield:

$$\begin{aligned}\frac{df_i^h}{dt} = {} & \sum_{r=1}^{n}\sum_{s=1}^{n}\left(\sum_{p=-H}^{H}\sum_{q=-H}^{H}\eta_{rs}^{pq}\Gamma_{rs}^{pq}(i,h)f_r^p f_s^q\right)-\sum_{s=1}^{n}\left(\sum_{p=-H}^{H}\sum_{q=-H}^{H}\eta_{is}^{pq}\Gamma_{is}^{pq}(k,h)f_i^p f_s^q\right)\\ & +\sum_{r=1}^{n}\sum_{\ell=1}^{m}\left(\sum_{p=-H}^{H}\sum_{q=-H}^{H}n_{i\ell}(t)g_\ell^q(v)\Lambda_{r\ell}^{pq}(i,h)f_r^p\right)\\ & -\sum_{\ell=1}^{m}\left(\sum_{p=-H}^{H}\sum_{q=-H}^{H}n_{i\ell}(t)g_\ell^q(v)\Lambda_{i\ell}^{pq}(k,h)f_i^p\right),\end{aligned} \tag{3.11}$$

where k represents the generic subsystem towards which particles belonging to subsystem i may switch. The above set of nonlinear equations provides the conceptual framework for the derivation of models as we shall see in the next section.

Finally, let us conclude this section stressing the importance of the definition of the mathematical operators which identify the dynamics: in the more general case the modelling aspects must take into account both the microscopic state of the interacting pairs, identified by p and q

in the proposed notation, and the macroscopic subsystems they belong to: i and j in the case of inner interaction, i and ℓ in the case of outer interaction, r, s and i in the case of switch. Special attention to the multiscale aspects and issues related to this kind of modelling are investigated in the last section.

4. Applications and Mathematical Problems

This section aims at showing, by means of two specific applications, how the general methodological approach presented in Section 3 can be technically applied. The examples are selected according to the personal author's experience and bias, while the aim consists in showing the treatment of the following issues:

- Selection of the activity variable and decomposition of the overall system into functional subsystems,
- Guidelines to model the interaction rates η_{ij}^{pq},
- Guidelines to model the interaction rates B_{ij}^{pq} and $G_{i\ell}^{pq}$, Γ_{rs}^{pq} and $\Lambda_{r\ell}^{pq}$.

The above analysis is developed at a preliminary qualitative stage, without dealing with specific analytic details, first for closed systems and subsequently for open systems. Further aim consists in identifying the optimization problems generated by the application of external actions and their costs. Let us stress again that this section has to be regarded as an introduction to issues that need a further development within a proper research program.

The first example we consider is the electoral competition within a democratic nation, where political parties and interest groups play a key role in the opinion dynamics. The voting process has been recently modelled with various mathematical tools related to different applications, many examples can be found in the literature: [48]-[57], while in the second example we model the phenomenon of job market mobility under the action of job advertisements. This topic has been analyzed with sociological and economic approaches, among others by [58] and [59].

4.1. *Electoral competition on the internet*

Let us now consider the modelling of an electoral competition, referred to a system corresponding to a nation subjected to the influence of internet media, such as blogs and online magazines. We decide to take into account this new kind of means of communication since they are becoming more and more influent in the socio-political panorama. Moreover social utilities like blogs, Facebook or Youtube involve a new active role of common citizens into the spread of information, in this sense the modelling differs from the modelling of classical media action that it is possible to find in [10]. In addition, this phenomenon is a good example of a multiscale type of interactions: common citizens can upload information on the internet that anyone can read, understand and moreover comment. As a consequence everyone can be part of an online community that creates debates on political themes, that can influence the policies of political parties and interest groups in general. The scientific community is nowadays paying attention to this kind of phenomena [60], [61] and some preliminary quantitative studies have appeared, like for example, [62].

Let us assume that u is a discrete variable, with $2p+1$ values, since it better represents the different levels of opinions, related to each group (subsystem). The domain interval can be referred to $[-1,1]$, where the two extreme values -1 and 1 represent, respectively, the two extreme political opinions *extreme left* and *extreme right*, 0 can be considered the *center opinion*:

$$I_u = \{u_{-p} = -1\,,\,...\,,\,u_0 = 0\,,\,...\,,\,u_p = 1\} \tag{4.1}$$

representing possible different levels of opinions.

Differently oriented political parties interact to win the elections, dealing with media information and interest groups. Following the approach proposed in the previous sections, the first step of the modelling consists in identifying the functional subsystems that compose the overall system. Accordingly, let us suppose to decompose the system into the following functional subsystems:

• Political right parties,

• Political left parties,

• Unions,

• White collars,

• Blogs and online magazines exerting an external action.

Let us focus on the above five subsystems to define a simple model suitable to describe the evolution of the system using a small number of parameters. We define the following discrete probability densities.

• $f_1^h(t)$, describes the distribution over the state of the subsystem ***political right parties***,

• $f_2^h(t)$, describes the distribution over the state of the subsystem ***political left parties***,

• $f_3^h(t)$, describes the distribution over the state of the subsystem ***unions***,

• $f_4^h(t)$, describes the distribution over the state of the subsystem ***white collars***.

• g_1^h, describes the distribution over the state of the subsystem ***online media*** which is not varying on time.

It is plain that this decomposition can be refined taking into account different parties (left, center and right oriented), additional interest groups and different online media (right, left and center oriented blogs and online newspapers).

Notice that the action of media is not considered to be certain since internal subsystems can actively decide to interact or not with the online media scenario, deciding which web site to connect to. However, the distribution over the state of the subsystem online media is considered not to change with time since, in general, the editorial line of media cannot vary in a short window of time. Since we are interested in relatively short-time phenomena, like the electoral competition, we do not take into account editorial line changes.

The second step consists in modelling the interaction terms, namely the matrix of the encounter rates η_{ij}^{pq} among unions, white collars and political parties, the parameters of the transition probability density, the inner-outer probability density regarding the external action of media and the switching rates ρ_{rs}^{pq}. Notice that $\ell = 1$, since we just deal with a single external agent.

Let us deal with the modelling of the encounter rate η_{ij}^{pq} according to the following assumption:

Hp. 1. $\eta_{ij} = \eta_{ji}$ *does not depend on the state of the interacting subsystems, but only on the* *interacting pairs.*

In particular, let us suppose that η_{ij} attains the largest value when the interactions takes places within the same subsystems since individuals belonging to the same subsystems are thought to be constantly in communication. Lower rates of interactions may be defined among individuals belonging to different subsystems. These rates must take into account the specific features of the subsystems itself: it is likely to happen that individuals belonging to the subsystem *union* interact more often with individuals belonging to the subsystem *left political parties* than to the subsystem *right political parties*; whereas individuals belonging to the subsystem *white collars* interact more often with individuals belonging to the subsystem *right political parties* than to the subsystem *left political parties*. From the economic point of view, these rates can be considered related to the concept of *lobbying*.

Let us now model the discrete transition probability density function. The modelling is developed according to the following assumption, consistent with the approach proposed in the previous section. The discrete probability density function, $B_{ij}^{pq}(h)$, represents the probability density that a candidate particle with state u_p of the population i falls in the state u_h of the test particle, after the encounter with a field particle with state u_q of the population j. We look for a discrete probability density function such that

$$\forall i,j, \quad \forall p,q: \qquad \sum_h B_{ij}^{pq}(h) = 1. \tag{4.2}$$

Let us introduce a *critical distance* d_c, such that, if encounters take place between individuals, whose opinions are closer than d_c, then the two parties get nearer according to their opinions (*dialectic case*) ; if encounters take place between individuals, whose opinions are more distant than d_c, then the two parties get farther according to their opinions (*non dialectic case*). The definition of the parameter d_c is essential in the modelling procedure since it characterize the degree of willingness to agree of a population of individuals. d_c may change according to the specific environment where subsystems are set.

Let us now model the external action over the inner subsystem i. According to the definition of the previous section, the influence of internet media ℓ over subsystem i can be modelled defining its action on time $n_{i\ell}$ and the inner-outer probability density $G_{i\ell}^{pq}$. Let us make the assumption that the action on time $n_{i\ell}$ varies according to subsystem i, while the inner-outer probability density $G_{i\ell}^{pq}$ varies according to the states p and q of the interacting particles only.

As far as $n_{i\ell}$ is concerned, let us assume that it is a decreasing function of t when $i = 3,4$, namely the subsystems Unions and White Collars, and that it is a constant function of t when $i = 1,2$, namely the subsystems Political Left or Right Parties, for example: $n_{1,2\ell}(t) = A$ and $n_{3,4\ell}(t) = Ae^{-t}$.

This assumption can be justified by the fact that political parties are structured to pay a great attention on electoral campaigns, devoting part of their organization to this duty, whereas other kinds of institutions may be affected by the news on the internet just for a short time, after which they tend to forget. A is a constant that measures the power of media action.

Focusing on the inner-outer probability density function, it can be defined analogously to the transition probability density function, by introducing a distance d_m between the state of the inner particle and the state of the external agent : if encounters take place between individuals, whose opinions are closer than d_m, then the two parties get nearer according to their opinions (*dialectic case*) ; if encounters take place between individuals, whose opinions are more distant than d_m, then the two parties get farther according to their opinions (*non dialectic case*). Notice that in the more general case, this new critical distance d_m may be different from the previous one d_c.

If we are dealing with the case in which switching phenomena are present in the system, the last step of the modelling process is the definition of the switching rates $\Gamma_{rs}^{pq}(i,h)$ and $\Lambda_{r\ell}^{pq}(i,h)$. Notice that if $r = i$ and s is an inner subsystem, the switching rate $\Gamma_{rs}^{pq}(i,h)$ describes the same action as the transition probability function and if $r = i$ and ℓ is an outer subsystem, the switching rate $\Lambda_{r\ell}^{pq}(i,h)$ describes the same action as the inner outer probability density function. In general, the definition of these rates must take into account both the subsystem which the interacting individuals belong to and their state. For example, it is possible to assume that for an individual is easier to switch from the subsystem unions to the subsystem political left parties than to the subsystem political right parties or that for an individual belonging to the subsystem white collars is easier to switch to the subsystem political right parties than to the subsystem political left parties. As a consequence, in this case, the first switching rates will be higher that the second ones.

However the considerations above do not take into account the change in state of the interacting particles: consequently, we may assume that, in the case of interactions between inner subsystems, there is no switch when the two states of the interacting pairs are such that $| p-q |> d_{cc}$, where d_{cc} is a new critical distance specifically defined for the switching rates. In this case $\Gamma_{rs}^{pq}(i,h) = 0$. The meaning of this assumption is that if the states of the interacting pair are too far, no interaction occurs and therefore no switch takes place. If $| p-q |< d_{cc}$ we suppose $h = [\frac{p+q}{2}]$. This formulation implies that in the cases above mentioned, a particle belonging to subsystem r with state p, encountering a particle belonging to subsystem s with state q, switch to subsystem i, going in the state $h = [\frac{p+q}{2}]$. Analogous reasonings can be done for $\Lambda_{r\ell}^{pq}(i,h)$.

Once all the rates and probability densities are defined, it is possible to obtain a system of ordinary differential equations for the discrete probability densities f_i^h and consequently to treat it with numerical simulations in order to investigate the evolutionary dynamics of the global system, see [10].

4.2. *Job mobility*

Let us now consider the modelling of a job market, among firms and institutions set in a global context. We suppose the job market to be externally influenced by the action of job advertisements.

We assume u to be a discrete variable, representing the different kinds of occupation, related to each subsystem, in this particular case representing companies. The domain interval $[-1,1]$ is defined as in (11), where I_u represents the different hierarchies of job duties, namely state -1 represents those jobs with less responsibility and state $+1$ those jobs with the highest responsibilities.

Differently oriented firms interact in order to establish their economic power on the global market. In this preliminary simple decomposition we can suppose that the goal of each firm is to attract workers sufficiently close in their tasks and duties to join their company and to dismiss those workers whose state is too distant from the global policy of the firm.

The specific model proposed in what follows is characterized by a small number of variables and very simple interaction functions. Let us suppose to decompose the system into the following subsystems:

- Firm 1, 2, 3

• Job advertisement exerting an external action.

Let us focus on the above four subsystems to define a simple model suitable to describe the evolution of the system using a small number of parameters. We define the following discrete probability densities.

• $f^h_{1,2,3}(t)$, describes the distribution over the state of the subsystems ***firm 1***, ***firm 2***, ***firm 3*** ,

• g^h_1, describes the distribution over the state of the subsystem ***job advertisements*** which is not varying on time.

As in the first example, the role of job advertisements is not considered to have a deterministic action, since generally the probability to find an advertisement is by far lower than 1. We assume the distribution of such advertisements not to change over time.

The second step consists in modelling the interaction terms, namely the matrix of the encounter rates η^{pq}_{ij} among firms, the parameters of the transition probability density $B^{pq}_{ij}(h)$, the inner-outer probability density regarding the external action of job advertisements $G^{pq}_{i\ell}(h)$ and the switching rates Γ^{pq}_{rs} and $\Lambda^{pq}_{r\ell}$. Notice that $\ell = 1$, since we deal with just a single external agent.

Let consider with the modelling of the encounter rate η^{pq}_{ij}, in contrast with the preceding application, we do not make the assumption that $\eta^{pq}_{ij} = \eta_{ji}$, namely it is reasonable to assume that the encounter rate depends also on the states of the interacting individuals: let us suppose that individuals interact more often with people with similar duties, having then a higher encounter rate, than with people with duties much different from theirs. Therefore, we make the assumption that η^{pq}_{ij} is a decreasing function of $\mid p-q \mid$.

Moreover let us suppose that η^{pq}_{ij} attains the largest value when the interactions takes places within the same firm ($i = j$) and lower values when interactions occur among individuals belonging to different firms ($i \neq j$). The modelling of η^{pq}_{ij} must take in consideration both these phenomena.

Let us now model the discrete transition probability density function. The modelling is developed according to the following assumption consistent with the approach proposed in the previous section.

The discrete probability density function, $B^{pq}_{ij}(h)$, represents the probability density that a worker with state u_p of the firm i falls in the state u_h, after the encounter with a worker with state u_q of the firm j. We look for a discrete function such that

$$\forall i,j, \quad \forall p,q: \qquad \sum_h B^{pq}_{ij}(h) = 1. \tag{4.3}$$

Again, let us introduce a *critical distance* d_c, such that, if encounters take place between workers, whose duties are closer than d_c, then the first individual gets a higher position within the firm (*career upgrade case*) ; if encounters take place between workers, whose duties are more distant than d_c, then the first individual gets a lower position within the firm (*career downgrade case*). The idea is to represent encounters among individuals who have more or less job responsibilities in different firms and, as a consequence, who may change their position being upgraded or downgraded within the same firm.

Notice that in this case the probability density function must also take into account the state u_p, namely the position, of the candidate particle within the firm. Since firms have a pyramidal organization it seems reasonable to imagine that the probability to change duty in the firm depends on the current position: it is more likely to be upgraded if in a low position, whereas it is more difficult if in a high position, since in a company there are many employees but few managers.

Let us now model the external action over the inner subsystem i. According to the definition of the previous section, the influence of job advertisements ℓ over subsystem i can be modelled defining its action on time $n_{i\ell}$ and the inner-outer probability density $G_{i\ell}^{pq}$.

As far as $n_{i\ell}$ is concerned let us assume that it is a decreasing function of t when $i = 1,2,3$, since advertisements are less efficient when time goes by:

$$n_{1,2,3\ell}(t) = Ae^{-t},$$

where A is a constant that measures the power of advertisements.

The last step of the modelling process consists in the definition of the switching rates $\Gamma_{rs}^{pq}(i,h)$ and $\Lambda_{r\ell}^{pq}(i,h)$. In general, when $r = i$ and $s = j$ the switching rates describe the same action as the inner transition probability function and when $r = i$ and ℓ is an outer subsystem the switching rates describe the same action as the inner outer probability density function. In particular, when a worker of firm r meets another worker of firm r and he/she goes into subsystem $i \neq r$, the worker is dismissed; when a worker of firm r meets a worker of firms $s \neq r$ and goes into subsystem $i = s$ the worker is hired. Workers may change firm also as a consequence of the external action, namely after finding good job advertisements, this is the case when $r \neq i$ and ℓ is the outer subsystem. These are the three cases in which the workers are effectively changing firm. In the other cases, they may change position within the same firm.

However the definitions above do not take into account the change in state of the interacting particles: consequently, we may assume that there is no switch when the two states of the interacting pairs are such that $| p-q |< d_{cc}$ and $p > q$, where d_{cc} is a new critical distance specifically defined for the switching rate. In this case the switching rates are equal to zero. Notice that this time there is switch just in the case in which d_{cc} is sufficiently high: the reason is that generally it is possible to move from one firm to the other when meeting high ranked workers of the firms, since these people are the one who can recruit or dismiss. In order to change company a worker with state p needs to meet another worker whose state is higher than p, able to recruit or dismiss him. In particular, when $r \neq s$ and $s = i$ there is switch and we are in the case of recruiting workers, when $r = s$ and $s \neq i$, instead, we are in the case of dismissing workers. If $| p-q |> d_{cc}$, $r \neq s$ and $s = i$ and $p < q$ we suppose $\Rightarrow h \geq p$, since we assume that in order to change company the candidate worker must receive an award in duty or at least not to be downgraded. If $| p-q |> d_{cc}$, $r = s$ and $s \neq i$ and $p < q$ we suppose $\Rightarrow h \leq p$, since we assume that in order to be hired, once fired, an individual accept any job is offered, even if of lower category. This formulation implies that in the above mentioned cases, a worker belonging to firm r with state p, encountering a worker belonging to firm s with state q and switching to firm i, goes into a state $h \geq p$ when recruited and into a state $h \leq p$ when dismissed.

Taking into account the previous definitions, it is possible to obtain a systems of ordinary differential equations where every operator is defined and consequently to treat it with numerical simulation in order to investigate the evolutionary dynamics of the global system.

5. A Deeper Analysis of Multiscale Issues

The preceding sections have shown how the mathematical methods offered by the kinetic theory for active particles can be properly developed to model complex systems within the general framework of behavioral economy. The contents have been focused both on mathematical tools and applications. As we have seen, an important aspect of the whole modelling approach is the decomposition of the overall system into functional subsystems, whose state is described by a specific activity variable linked to a discrete probability density. The mathematical structure, which acts as a background paradigm for the derivation of models, according to Eq.(3.9), can be written in compact form as follows:

$$\frac{df_i^h}{dt} = J_i^h[f] = \sum_{j=1}^{n} (J_{ij}^h[f] + Q_{ij}^h[f,g]), \tag{5.1}$$

where the terms J_{ij}^h and Q_{ij}^h refer, respectively, to interactions among active particles and between active particles and the outer environment. Therefore, interactions are modelled at the microscopic scale only (taking into account both the microscopic interaction within and among functional subsystems), while systems, in several significant cases, can reciprocally act as a whole. These interactions can also imply aggregation or decomposition of systems and macroscopic interactions among functional subsystems. In this section we propose, referring to the literature in the field, e.g. [63] and [64], a first attempt of the modelling of these multiscale processes.

The reasonings proposed in what follows refer to the mathematical structure (5.1), while it can be technically generalized to the case of the structures (3.11), which involve mobility of active particles. To give a complete overview of the issue, let us first consider the continuous case, to switch subsequently to the discrete case. Let us introduce the equation

$$\begin{aligned}\partial_t f_i(t,u) = \sum_{j=1}^{n} \int_{\mathbf{R}\times\mathbf{R}} \eta_{ij}(u_*,u^*) B_{ij}(u_*,u^*;u) f_i(t,u_*) f_j(t,u^*)\, du_*\, du^* \\ - f_i(t,u) \sum_{j=1}^{n} \int_{\mathbf{R}} \eta_{ij}(u,u^*) f_j(t,u^*)\, du^*\,,\end{aligned} \tag{5.2}$$

where u_* and u^* represent, respectively, the state, considered as a continuous variable, of subsystems i and j. Let us introduce the term $K_j = K_j[f_j](u)$ which identifies a macroscopic operator over f_j, which models the macroscopic action on the distribution f_j. This operator may be supposed an integral operator, since it must capture the macroscopic features of the distribution. Subsequently, the total derivative of f_i must take into account this new term:

$$\frac{\partial f_i}{\partial t} + \sum_{j=1}^{n} \frac{\partial}{\partial u}(K_j[f_j] f_i) = \sum_{j=1}^{n} (J_{ij}[f] + Q_{ij}[f,g]). \tag{5.3}$$

In some practical cases the action K_j is identified by moments weighted by f_j.

Let us now introduce the guidelines for the modelling of discrete systems, for simplicity, in the following paragraphs we do not take into account the microscopic external action represented by the term $Q_{ij}[f,g]$. As in the continuous case, the action of j over i is modelled by means of the macroscopic operator K_j which depends on the discrete moments $E_j^p[f_j]$ of the discrete distribution f_j. Notice that, in general, this action may or may not depend on the state h.

Let us represent the macroscopic action on the system by means of a source term S_i^h. Since this term can represent a gain or a loss in the state h, we may introduce the notation S_i^{h+} to describe the gain term and S_i^{h-} describing the loss term, where

$$S_i^h = S_i^{h+} - S_i^{h-}. \tag{5.4}$$

The state h can gain or lose particles due to to the macroscopic action. Notice that this term is not a binary interaction term but a drift term.

Let us figure that the gain in the state h of the subsystem i involves processes at the state $h-1$ due to the macroscopic actions exerted by the subsystems $j = 1,...,n, j \neq i$, namely we assume that the macroscopic action on subsystem i is exerted by subsystems different from i:

$$S_i(h-1 \to h) = \sum_{j=1, j\neq i}^{n} K_j^{h-1}[E_j^p] f_i^{h-1}. \tag{5.5}$$

This is a progressive action which leads to a shift of particles from the state $h-1$ towards the state h. For symmetry, it is possible to have a shift into the state h thanks to those particles moving from the state $h+1$ towards h,

$$S_i(h+1 \to h) = \sum_{j=1, j\neq i}^{n} K_j^{h+1}[E_j^p] f_i^{h+1}. \tag{5.6}$$

Therefore $S_i(h-1 \to h) + S_i(h+1 \to h) = S_i^{h+}$.

Following an analogous reasoning, the loss term is given by

$$S_i^{h-} = \sum_{j=1, j\neq i}^{n} K_j^{h}[E_j^p] f_i^{h}, \tag{5.7}$$

where the action of the shift from state h to another state is modelled. As a consequence of the given definitions, the total microscopic and macroscopic variation of particles is represented by the following equation, where both the microscopic and macroscopic interaction and action terms are present:

$$\frac{df_i^h}{dt} = \sum_{j=1, j\neq i}^{n} (K_j^{h-1}[E_j^p] f_i^{h-1} + K_j^{h+1}[E_j^p] f_i^{h+1} + K_j^{h}[E_j^p] f_i^{h}) + \sum_{j=1}^{n} J_{ij}^h[f] \tag{5.8}$$

Since we are in the discrete case it is important to remark that the bounds of the domain interval of the states cannot be overtaken and that the total sum of gain and loss terms must be equal to 0, namely the total number of particle is constant.

In conclusion, let us give some brief remarks related to show how this general framework can be adapted to the application described in the previous section. Just to give a very general insight, notice that both phenomena, namely diffusion of political opinions subjected to social utilities influence and job mobility are clearly affected by macroscopic events, that cannot be modelled only by binary interactions. The effects of news, web job advertising or financial crisis that provoke massive firing or hiring are macroscopic drift forces that must be taken into account in the modelling procedure. In order to propose coherent applied models, it is necessary to model carefully the macroscopic terms K_j and to link the macroscopic modelling of interactions to the whole system choosing which discrete moments of the distribution f_j have to be considered and introduced in the mathematical equations. According to the macroscopic moments of each distribution f_j the gain and loss terms vary, for example the mean of the distribution of the states of different groups of interests, in the first example, or firms, in the second one, can affect the macroscopic interactions modelled by the gain and loss terms.

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TOWARDS A MATHEMATICAL THEORY OF COMPLEX SOCIO-ECONOMICAL SYSTEMS BY FUNCTIONAL SUBSYSTEMS REPRESENTATION

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Abstract. This paper deals with the development of a mathematical theory for complex socio-economical systems. The approach is based on the methods of the mathematical kinetic theory for active particles, which describes the evolution of large systems of interacting entities which are carriers of specific functions, in our case economical activities. The method is implemented with the concept of functional subsystems constituted by aggregated entities which have the ability of expressing socio-economical purposes and functions.

1. **Introduction.** This paper deals with the development of a mathematical theory to model complex socio-economical systems, where individual behaviors and interactions may play a significant role on the evolution of the system.

The mathematical approach takes advantage of the kinetic theory for active particles [7] already applied in various fields of life sciences, e.g. modelling multicellular systems [9], [11], and social behaviors of interacting individuals [12], [13], [15]. This mathematical theory describes the evolution of the probability distribution over the microscopic state, called activity, of several interacting entities called active particles. The equation which models the evolution is derived by a conservation balance in the elementary volume of the space of the microscopic states, where the inlet and outlet flows are determined by interactions among active particles.

An additional essential tool for the theory is the concept of modules proposed by Hartwell [22] for the interpretation of complex biological systems. Briefly, a module is viewed as an aggregate of entities specialized to develop a well defined function referred to the system under consideration. A modification of the objectives of the investigation may possibly modify also the structure of the modules.

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This paper develops the above concepts introducing the idea of ***functional subsystem*** that refers to both the functions expressed by the socio-economical system, which is the system under consideration, and the observation and representation scale used in the mathematical modelling process. The whole system is composed by several interacting sub-systems, each related to a specific socio-economical function. The time-evolution of the system is described by differential equations delivered by a suitable development of the mathematical kinetic theory for active particles [7].

Before approaching the afore mentioned topics, it is worth analysing some conceptual aspects on the interaction between applied mathematics and social sciences. The main problem consists in investigating how the ***qualitative interpretation*** of social reality, that is delivered, including the interpretation of empirical data, by research activity in social sciences, can be transferred into a suitable ***quantitative description*** through mathematical equations. The above target needs the development of a dialogue between methods and traditions of two different disciplines with the additional difficulty of dealing with living systems, which have the ability to think and, as a consequence, to react to external actions, without following rules constant over time. This dialogue, however difficult, is necessary to a deeper understanding of the so called ***behavioral economics***, where deterministic rules may be stochastically perturbed by individuals behaviors. An additional difficulty to be taken into account is that individual behaviors not only show random fluctuations, but may be substantially modified by external environments, from depressive or panic situations to over-optimistic attitudes induced by misleading propaganda.

This paper aims at developing a first approach to the design of a mathematical theory focused on modelling the evolution in time of interacting socio-economical systems. The contents of the paper are organized in six sections, which follow the above introduction.

Section 2 first analyses some scaling issues related to the representation of economical systems and consequently proposes the concept of ***functional subsystem*** as an essential paradigm to develop the mathematical theory.

Section 3 shows how complex economical systems can be viewed as a network of several interacting functional subsystems, each of them having the ability to express economical functions and purposes. The proposed mathematical representation defines a probability distribution over their functional states. Models in which functional subsystems express deterministically their functions, should be considered as particular cases of the more general stochastic description.

Section 4 is devoted to the application of the above mentioned mathematical method to a variety of socio-economical systems, chosen among the ones described in Section 2. The relation of the approach proposed in this paper with respect to the existing literature is critically analysed in this section.

Section 5 deals with the derivation of a suitable mathematical structure, based on the kinetic theory for active particles, which leads to evolution equations over the probability distribution function introduced in Section 3. The contents refer to networks of interacting subsystems, both in the case of isolation from the outer world (***closed systems***), and in the case of stochastic interactions with the outer environment (***open systems***). Both mathematical structure, for closed and open systems, act as a paradigm for the derivation of models.

Section 6 shows how to implement this mathematical theory to derive a model by means of a methodological modelling approach to microscopic interaction. Subsequently, the method is applied, in Section 7, to the derivation of a model of competition for a secession.

Section 8 critically analyses some specific applications and provides an overlook on research perspectives, also focused on suitable developments of the mathematical structures proposed in Section 5, in order to enlarge the variety of socio-economical systems which can be described by mathematical models.

2. **From scaling to the concept of functional subsystems.** The first step towards the mathematical modelling of the behavior of real world systems consists in defining the proper observation and representation scale, for short the ***scaling problem***.

This issue is well documented in the physics of real systems [32], however the reasoning applies also to the case of socio-economical systems, with the additional difficulty of assessing the various and technically different components of the system. Moreover, dealing with socio-economical systems also needs to overcome the difficulties related to dealing with living systems, as individual behaviors can play a remarkable role in the evolution of economical phenomena.

A general rule to identify the appropriate observation scale when dealing with the inert matter in a real world system is the following: the microscopic scale identifies the smallest entities composing the system, the macroscopic scale, instead, refers to quantities which can be observed and measured. These observable quantities correspond to local averages over the states of interacting microscopic entities.

As an alternative, used in the sequel of this paper, it is possible to deal with the ***statistical (kinetic) representation***, where the state of the whole system is described by a suitable probability distribution over the microscopic state of the interacting elements. Macroscopic observable quantities are recovered by means of weighted moments of the distribution function.

In the case of living systems the assessment of what is small and what is large appears to be quite conventional and often related to the type of analysis that is developed. Moreover, living systems, differently from inert matter systems, are characterized by their ability of organizing specific strategies according to well defined objectives. This aspect is documented in the paper by Hartwell et al. [22] and developed in different fields of biology [10].

A preliminary step is the identification of microscopic and macroscopic scales. This aspect suggests to develop the definition of functional subsystems. The reference scaling in Economics is delivered by the concept of ***microeconomics*** and ***macroeconomics***. Although the above categorization shows some flexibility, it is well accepted that microeconomics refers to the interaction between economical agents, say individuals, households, enterprizes, and so on, that generate economical interactions and processes in the market. On the other hand, macroeconomics refers to aggregated economical processes, e.g. national income, unemployment, inflation, investment, international trade, which are the output both of the above interactions at the level of microscopic agents and of running of a country, e.g. macroeconomics is highly correlated with the development of economic policy and strategies.

The above reasonings are specifically referred to three examples of socio-economical systems which can be examined through the mathematical approach proposed in this paper. These applications are here analysed with both theoretical and applied tools.

The examples proposed in the following subsections refer to socio-economical phenomena, which have been recently analysed in Economics, even if from a different mathematical point of view. The innovative mathematical structure developed in this paper is proposed in order to give a unique and general mathematical approach to all these phenomena. A similar approach to socio-economical phenomena can be found in [19], [20], [25], [26], [33], where the influence of the social context on economical decisions is deeply considered and studied.

In particular, going through the economic literature of these last years, it is possible to find many attempts to explain socio-economical contemporary issues: as far as the example in Subsection 2.1 is concerned, it is possible to refer to [4], [5], [6], [16], [30], [31], where the mechanisms which rule country separations and even civil conflict secessions are properly dealt with.

Referring to the example of Subsection 2.2, concerning the democratization of a country, it is possible to refer to [2], [3], [23], [24], where a model has been proposed to explain how democracies or dictatorships can survive under a popular consensus or why the right of vote can be extended to all citizens. Additional valuable references are offered by Rubinstein [28].

Subsection 2.3 refers to *welfare politics*, focusing on the mechanisms according to which it is ruled by very general aspects of political economy. This issue is also dealt with in various papers [12], [13], [15], that apply methods of the discrete kinetic theory and stochastic games to social competition mathematical models. Methods derived from statistical mechanics are used to analyse political, economical, and social competitions phenomena [17], [18], [21], [29]. The use of multiple interaction games is proposed in papers[27], [8].

Finally, the last subsection proposes some additional reasonings on various aspects concerning further interpretations of the above socio-economical systems.

2.1. **Competition for a secession.** The overall system is a nation which, as visualized in Table 2.1, is decomposed into two or more subsystems identified by regional interest groups, which express, through specific actions taken by either their political parties or their peculiar interest groups, their attitude towards a process of secession by expressing a function u, which takes negative or positive values: when a certain region is expressing a positive value of u, then it is pro-secession, when it is expressing a negative value of u, then it is against it. Again, the absolute value of u measures the intensity of the expressed function.

Global nation			
Region1		Region2	
political parties	interest groups	political parties	interest group
$u > 0$ - pro	$u < 0$ - against	$u > 0$ - pro	$u < 0$ - against

2.1. Competition for a secession

2.2. **Democratization of a dictatorship.** The overall system is decomposed, as visualized in Table 2.2, into four subsystems: Dictator, Ministers, Parliament, and Citizens. All of them express, by technically different channels, a socio-economical function u, either to support the dictatorship, or to contrast it through dissidence. This function takes negative values when it is supporting, and positive values when it is contrasting. The absolute value of the function measures the expressed intensity of support or contrast.

Dictatorship							
Dictator		Ministers		Parliament		Citizens	
support	dissid.	support	dissid.	support	dissid.	support	dissid.
$u < 0$	$u > 0$	$u < 0$	$u > 0$	$u < 0$	$u > 0$	$u < 0$	$u > 0$

2.2.Democratization of a dictatorship

2.3. **Welfare politics.** The overall system is a nation which, as visualized in Table 2.3, is decomposed into subsystems identified by different interest groups, i.e. those groups which have some specific economical power within the political economical equilibria for a given country, which compete in order to determine "right" or "left" political economical decisions.

Country							
Firms		Banks		Political parties		Unions	
Left	Right	Left	Right	Left	Right	Left	Right
$u < 0$	$u > 0$	$u < 0$	$u > 0$	$u < 0$	$u > 0$	$u < 0$	$u > 0$

2.3. Welfare politics

2.4. **Additional reasonings.** The above examples want to show how real socio-economical phenomena can be interpreted and decomposed using the concept of functional subsystems. Each example shows a hierarchical decomposition from the global system to different subsystems, expressing the social activity u. This variable conceptually varies form one case to the other: this means clearly that the specific meaning of u is strictly related to the phenomenon under consideration.

Even the structure of the decomposition of the global system varies corresponding to the different examples shown above; this depends on which different aspects of the phenomenon it will be interesting to focus on.

Keeping this in mind, the afore-mentioned decomposition can be an applied starting point to understand the development of the mathematical framework which describes the subsystem itself. This framework analyses the evolution of the socio-economical variable u, as an effect of interactions among the different subsystems of the global structure. The different social actors of a dictatorship, (the dictator, the ministers, the members of the parliament, the citizens themselves), for example, usually interact to develop a certain kind of society. The consequence of this interaction can lead to the fall or the enhancement of the dictatorship itself: this process is described by the evolution of the variable u, introduced over the above schemes. Moreover, the same reasoning can be applied to the phenomena of secession or to political economical competition inside a certain country: again it is possible to describe whether a nation breaks up into many regions or whether a country is oriented toward liberal or toward socialist economy, as a consequence of the interactions between the subsystems identified in the corresponding examples.

Finally, it is important to remark that the above systems are viewed as closed with respect to the external world. If the system is open, interactions with the outer environment have to be specifically modelled.

3. **Socio-economical systems as interacting functional subsystems.** The overall socio-economical system is viewed, according to the analysis of Section 2, as a network of interacting functional sub-systems. Their representation is based on the assumption that each of them has the ability to express a specific function.

Bearing this in mind consider a network of n interacting functional subsystems whose function is identified by the variable $u \in \mathbb{R}$, where the value $u = 0$ separates the positive and negative valued ***functions*** expressed by each subsystem. The overall state of the system is described by the probability distributions:

$$f_i = f_i(t,u) : \quad [0,T] \times \mathbb{R} \to \mathbb{R}^+ \quad f_i \in L_1(\mathbb{R}), \tag{1}$$

where the subscript refers to the i^{th} subsystem, and where for each of them:

$$\lim_{|u|\to\infty} f_i = 0, \qquad \int_{\mathbb{R}} f_i(t,u)\, du = 1, \quad \forall\, t \geq 0, \tag{2}$$

Each f_i has the structure of a probability density such that

$$\int_a^b f_i(t,u)\, du = P(t; u \in [a,b]) \tag{3}$$

is the probability that the i^{th} subsystem expresses a function in the range $[a,b]$.

Remark 1. The variable u has to be regarded as a dimensionless quantity in units of a suitable reference quantity properly selected for each particular system. The fact that u ranges over an unbounded domain prevents the *a priori* identification of its upper and lower bounds, which in some cases may be technically difficult. On the other hand, if the upper or lower bounds are identified, the domain of u can be assumed as follows: $u \in D_u = [-1,1]$.

Remark 2. Let us consider the case where u is a scalar variable. In other words, the decomposition of the whole system into functional subsystems is organized in such a way that the expressed activity is a scalar. Therefore, i and u entirely describe the subsystem.

Macroscopic quantities can be computed, under the assumption $u^n f_i \in L_1(\mathbb{R})$, for $n = 1, 2$, by weighted moments corresponding, for instance, to mean value, energy and variance, respectively:

$$L_i(t) = \int_{-\infty}^{\infty} u\, f_i(t,u)\, du, \tag{4}$$

$$E_i(t) = \int_{-\infty}^{\infty} u^2\, f_i(t,u)\, du, \tag{5}$$

$$\sigma_i(t) = \int_{-\infty}^{\infty} \big(u - L_i(t)\big)^2 f_i(t,u)\, du. \tag{6}$$

Remark 3. The analysis of models, i.e. the solution of mathematical problems related to the application of models to real systems analysis, takes into account, at least in some cases, the initial value of the quantities L_i and E_i:

$$L_{i0} = L_i(t=0), \qquad E_{i0} = E_i(t=0).$$

In fact, interactions between functional subsystems can depend also on their initial properties.

The variable u refers to the specific function of each functional subsystem. In other words, the functional subsystem is referred to one function only; this means that large systems are decomposed, to reduce complexity, into several subsystems each expressing a scalar function. The possibility of dealing with subsystems expressing more than one function and the way to decompose them, is analysed in the last section referring to the examples proposed in Section 2.

The above assumption that functions are identified by continuous variables has the advantage that lower and upper bounds do not need to be fixed a priori, while it has to be acknowledged that discrete variables have the advantage that empirical data obtained by measurements can be related precisely to ranges of variability. This argument is posed in various papers dealing with modelling social dynamics [12], [13].

A brief indication on the use of discrete variables is given for the sake of completeness. In this case, the ***function*** is identified by a discrete variable

$$I_u = \{u_{-p} = -1, \dots, u_0 = 0, \dots, u_p = 1\},$$

where the description of the state of each subsystem is delivered by the discrete probability density

$$f_i^j = f_i^j(t) : \quad [0,T] \times \mathbb{R} \to \mathbb{R}_+^n , \qquad \sum_{j=1}^{p} f_i^j(t) = 1 , \quad \forall\, t \geq 0 , \tag{7}$$

for $i = 1, \ldots, n$ and $j = -p, \ldots, p$.

First and second order moments are given by sums as follows:

$$L_i(t) = \sum_{j=-p}^{p} u_j\, f_i^j(t) , \qquad E_i(t) = \sum_{j=-p}^{p} u_j^2\, f_i^j(t) . \tag{8}$$

Remark 4. The use of a discrete distribution is motivated by the technical difficulty to identify the state of functional subsystems by a continuous variable, while suitable ranges of the state, obtained by discretization, allow in a simpler manner the socio-political collocation.

Remark 5. Specific models may possibly include states $j = -p$ and/or $j = p$, where at the initial time one has $f_i^{-p} = 0$ and/or $f_i^p = 0$. Subsequently the dynamics may lead to an evolution where these states are characterized by some probability higher than zero.

Let us anticipate, with respect to the detailed analysis of the next section, that the mathematical structure candidate to derive specific models for closed systems can be formally written as follows:

$$\partial_t f_i = J_i[f](t,u) \quad i = 1, \ldots, n , \tag{9}$$

where J_i is a suitable operator acting over the whole set of probability distributions $f = \{f_i\}$.

It is worth stressing that the formal structure (9), whose detailed expression will be given in the next section, acts as a paradigm for the derivation of specific models based, as we shall see, on a detailed modelling of the interactions in the network.

The above structure is meaningful for a close system, that is in absence of interactions with the outer environment. The modelling of open systems needs a representation of the outer functional subsystems by their actions on the inner system as follows:

$$A_{ji} = \varepsilon_{ji}\, g_j(t,v) : \quad [0,T] \times \mathbb{R} \to \mathbb{R}^+ \quad g_j \in L_1(\mathbb{R}) , \tag{10}$$

where ε_{ji} models the intensity over the ith subsystem due to the action of the jth agent represented by the probability density $g_j(t,v)$, where v is the expressed function and

$$\lim_{|u| \to \infty} g_j = 0 , \qquad \int_{\mathbb{R}} g_j(t,u)\, du = 1 , \quad \forall\, t \geq 0 , \tag{11}$$

In this case the formal structure of the equation can be written as follows:

$$\partial_t f_i = J_i[f](t,u) + Q_i[f,g](t,u) , \qquad i = 1, \ldots, n , \tag{12}$$

where $g = \{g_j\}$.

Structures (9) and (12) will be derived for a system decomposed into a fixed number of subsystems. On the other hand, one also has to consider, as we shall see in the last section, the case of subsystems which have the ability to aggregate with other subsystems or to decompose each of them into two or more subsystems.

The corresponding structure for systems with discrete functions is written as follows:

$$\frac{d}{dt} f_i^j = J_{ij}[\mathbf{f}](t,u) + Q_{ij}[\mathbf{f},\mathbf{g}](t,u) \quad i = 1, \ldots, n , \tag{13}$$

where $\mathbf{f} = \{f_{ij}\}$ and $\mathbf{g} = \{g_{ij}\}$.

4. **Representation of functional subsystems.** This section applies the representation rules proposed in Section 3 to the mathematical description of real socio-economical systems decomposed into a network of several interacting subsystems. Three specific systems are selected among those reported in Section 2 so that the application effectively refers to the real world.

4.1. **Representation of "Competition for a secession".** This phenomenon is modelled through the following decomposition: the global system is a nation, decomposed into two or more regions. Each region is decomposed into subsystems identified by different interest groups and political parties, which are to be assumed as those institutions having such an economical and political power as to enforce a secession process. Each group can express a variable u in favor or against secession; again let us suppose that positive values represent support to secession and negative ones opposition to it. The role of the distribution f_i is the same as before, indicating the probability that the interest group, labelled by i, is pro or against secession. $L_i(t)$ is the mean value of f_i, representing the mean desire of secession (***activation***) for the subsystem i. $E_i(t)$ is the mean energy with which the degree of secession desire is expressed (***activation energy***), and $\sigma_i(t)$ is the distribution variance.

4.2. **Representation of "Democratization of a dictatorship".** We model this phenomenon by decomposing the dictatorship-state into four functional subsystems, each of them representing a certain political institution: the dictator itself and its close "entourage", the ministers of the government, the parliament and the common citizens. Each subsystem expresses a variable $u \in \mathbb{R}$ which represents its attitude towards the dictatorship, namely whether the subsystem supports or opposes to it. Let us suppose that positive and negative values represent respectively dissidence and support. The overall state of the subsystem is described by the probability distribution f_i over u, which indicates the probability that one of the subsystems is in a state u at time t.

The macroscopic quantities defined in the previous section can assume a practical meaning. $L_i(t)$ is the mean value for the distribution f_i representing the mean degree of dissidence or support (***activation***) for the $i-th$ subsystem, $E_i(t)$ is the mean energy with which dissidence or support is expressed (***activation energy***), and $\sigma_i(t)$ is the variance of the distribution, from the mean $L_i(t)$.

4.3. **Representation of "Competition for a welfare politics".** This example refers to the decomposition of a state nation into different interest groups, which are supposed to be the political and economical powers able to compete for a certain political economical policy. Each group expresses a variable u which can be left (if negative valued) or right (if positive valued) oriented in terms of welfare state decisions. As before, $L_i(t)$ is the mean degree of political orientation (***activation***) for each subsystem, $E_i(t)$ the mean energy with which the political orientation is expressed (***activation energy***) and $\sigma_i(t)$ the variance from the defined mean, for each subsystem i.

5. **Mathematical structures for closed and open systems.** This section deals with the derivation, within the framework of the kinetic theory for active particles, of mathematical structures to model systems which do not (or do) interact with the outer world. These structures acts, as already mentioned, as a general paradigm for the derivation of models referred to specific socio-economical systems.

The first two next subsections deal with the derivation of the above mentioned frameworks, while the last section offers some additional reasoning on general issues to be critically analysed in the last section. The derivation technically refers to the recent papers by Bertotti and Delitala [13], [14] concerning models with discrete valued states of the variable u.

5.1. **Closed systems.** This section deals with the mathematical structure which can be used to describe a socio-economical *closed system* regarded as a set of interacting functional subsystems which do not interact with any system of the outer environment. In other words, we look for a mathematical structure of the type reported in Eq. (9).

Moreover, we focus on systems in which the number n of subsystems is constant: this means that we are not dealing with destructive or aggregative events. Interactions are supposed to be binary. Generalizations are considered in the last section.

The mathematical framework to describe microscopic interactions between two subsystems can be described by means of two different functions:

• the ***encounter rate*** $\eta_{ij}(u_*, u^*)$, that describes the rate of interactions between a subsystem i with state u_* and a subsystem j with state u^*.

• the ***transition probability density*** $B_{ij}(u_*, u*, u)$, that describes the probability density that a subsystem i with state u_* falls into the state u after the interaction with a subsystem j with state u^*.

The term $B_{ij}(u_*, u^*, u)$, according to its property of probability density, satisfies the following condition:

$$\forall u_*, u^* \in \mathbb{R}\,, \quad \forall i, j \,:\quad \int_{\mathbb{R}} B_{ij}(u_*, u^*, u)\, du = 1. \tag{14}$$

It is now possible to derive the equation which defines the evolution of the probability distribution over the microscopic state by a balance equation of the inlet and outlet flows in the elementary volume $[u, u + du]$ in the space of the microscopic states. Technical calculations yield:

$$\begin{aligned}
\partial_t f_i(t,u) &= G_i[f](t,u) - L_i[f](t,u) \\
&= \sum_{j=1}^{n} \int_{\mathbb{R}\times\mathbb{R}} \eta_{ij}(u_*, u^*) B_{ij}(u_*, u^*; u) f_i(t, u_*) f_j(t, u^*)\, du_*\, du^* \\
&- f_i(t,u) \sum_{j=1}^{n} \int_{\mathbb{R}} \eta_{ij}(u, u^*) f_j(t, u^*)\, du^*\,,
\end{aligned} \tag{15}$$

where G_i and L_i denote, respectively, the inflow and outflow, at time t, into (and out) the elementary volume $[u, u+du]$ of the space of the states for each population. This equation can describe a real economical systems after the identification of the key functions η and B.

Remark 6. As a consequence of (15), provided that f_i is integrable, the following equation holds:

$$\partial_t \int_{\mathbb{R}} f_i(t,u) du = 0\,, \quad \forall\, t \geq 0\,. \tag{16}$$

The above property implies that the zero-th order moment is preserved in time:

$$\int_{\mathbb{R}} f_i(t,u)\, du = \int_{\mathbb{R}} f_i(0,u)\, du\,, \tag{17}$$

while, due to interactions, the higher order moments evolve in time.

As far as discrete systems are concerned, taking into account a discretization of the socio-economical variable u such that $I_u = \{u_1, ..., u_h, ..., u_H.\}$, the microscopic interaction between two subsystems is described by the following discrete functions:

• the ***encounter rate*** η_{ij}^{pq}, which depends on the interacting subsystems i and j and on the socio-economical states, p and q respectively, of the interacting particles.

• the ***transition probability density*** $B_{ij}^{pq}(h) = B_{ij}(u_p, u_q; u_h)$, that describes the probability density that a candidate particle with state u_p of the subsystem i falls into the state u_h after the interaction with a field particle of the subsystem j with state u_q.

The transition probability density function is such that:

$$\forall i, j, \quad \forall p, q : \qquad \sum_{h=1}^{H} B_{ij}^{pq}(h) = 1. \tag{18}$$

In this case the balance equation describing the evolution of the probability distribution over the microscopic state leads to the following system of $n \times H$ ordinary differential equations:

$$\frac{df_i^h}{dt} = \sum_{j=1}^{n} \left(\sum_{p=1}^{H} \sum_{q=1}^{H} \eta_{ij}^{pq} B_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=1}^{H} \eta_{ij}^{pq} f_j^q \right). \tag{19}$$

5.2. **Open systems.** Let us consider the derivation of a mathematical structure for socio-economical ***open systems***, that interact also with the outer environment. This means that, considering n interacting functional subsystems, the i-th subsystem interacts with external agents identified with the subscript $h = 1, ..., m$, where m is constant.

The external agent h has the ability to influence the socio-economical state u of the subsystem i, through a particular action identified by the variable v. The analysis developed in what follows is based on the assumption that the action of the outer agents can be modelled as follows:

$$\varepsilon_{hi}(t)\, g_h(v)\,, \tag{20}$$

where $\varepsilon_{ih} = \varepsilon_{ih}(t)$ is the intensity, that can depend on time, with which the agent h acts on the subsystem i; and $g_h(v)$ is the probability density associated to the variable v that characterizes the action of outer system.

Remark 7. Each external agent is regarded as a specific functional sub-system with the ability to interact with the functional sub-systems of the inner system. Both terms ε_{hi} and g_h are supposed to be given functions of their arguments. Specific examples are given in the next section.

The equation defining, for each functional sub-system, the evolution of the probability distribution over the microscopic state can be derived, as in the previous section, by a balance equation of the inlet and outlet flows in the elementary volume $[u, u + du]$ of the space of the microscopic states.

The mathematical framework to describe the microscopic interactions between a subsystem i and an external agent h needs the following interaction terms:

• The ***inner-outer encounter rate*** $\mu_{hi}(v_*, u^*)$ which describes the rate of interactions between an external agent h with state v_* and a subsystem i with state u^*.

• The ***inner-outer transition probability density*** $G_{hi}(v_*, u*, u)$, which describes the probability density that a subsystem i with state u^* falls into the state u after an interaction with an external agent h with state v_*.

The interaction term $G_{hi}(v_*, u^*, u)$ satisfies the following condition:

$$\forall v_*, u^* \in \mathbb{R}\,, \quad \forall h, i \qquad \int_{\mathbb{R}} G_{hi}(v_*, u^*, u)\, du = 1\,. \tag{21}$$

Technical calculations analogous to those we have seen in the preceding section yield:

$$\begin{aligned}\partial_t f_i(t,u) &= \sum_{j=1}^{n} \int_{\mathbb{R}\times\mathbb{R}} \eta_{ij}(u_*,u^*) B_{ij}(u_*,u^*;u) f_i(t,u_*) f_j(t,u^*)\, du_*\, du^* \\ &- f_i(t,u) \sum_{j=1}^{n} \int_{\mathbb{R}} \eta_{ij}(u,u^*) f_j(t,u^*)\, du^* \\ &+ \sum_{h=1}^{m} \int_{\mathbb{R}\times\mathbb{R}} \varepsilon_{hi}\, \mu_{hi}(v_*,u^*) G_{hi}(v_*,u^*;u) g_h(t,v_*) f_i(t,u^*)\, dv_*\, du^* \\ &- f_i(t,u) \sum_{h=1}^{m} \int_{\mathbb{R}} \varepsilon_{hi}\, \mu_{hi}(v_*,u) g_h(t,v_*)\, dv_*\,. \end{aligned} \tag{22}$$

In the case of discrete systems the subscript indicating the external agent h is replaced by the subscript l, since h indicates the discretization step. In this case we assume that the action of the outer agent can be modelled as $\varepsilon_{i\ell}\, g_l(v)$, where v is a discrete variable. The mathematical framework to describe the microscopic interactions between a subsystem i and an external agent ℓ needs the following discrete interaction terms:

- The ***inner-outer encounter rate*** $\mu_{i\ell}$ that describes the rate of interactions between an external agent l and a subsystem i and does not depend on the socio-economical state of the interacting particles.
- The ***inner-outer transition probability density*** $G_{i\ell}^{pq}(h) = G_{i\ell}^{pq}(v_q,u_p,u_h)$, that describes the probability density that a candidate particle with state u_p of the subsystem i falls into the state u_h after the interaction with a field particle of the external agent ℓ with state v_q.

The interaction term $G_{i\ell}^{pq}(v_q,u_p,u_h)$ satisfies the following condition:

$$\forall p,q, \forall i,\ell: \quad \sum_{h=1}^{H} G_{i\ell}^{pq}(h) = 1. \tag{23}$$

Technical calculations similar to the preceding one yield:

$$\begin{aligned}\frac{df_i^h}{dt} &= \sum_{j=1}^{n} \left(\sum_{p=1}^{H}\sum_{q=1}^{H} \eta_{ij}^{pq} B_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=1}^{H} \eta_{ij}^{pq} f_j^q \right) \\ &+ \sum_{\ell=1}^{m} \left(\sum_{p=1}^{H}\sum_{q=1}^{H} \varepsilon_{i\ell}\mu_{i\ell}^{pq} G_{i\ell}^{pq}(h) g_\ell^q f_i^p - f_i^h \sum_{q=1}^{H} \varepsilon_{i\ell}\mu_{i\ell}^{pq} g_\ell^q \right). \end{aligned} \tag{24}$$

5.3. **Additional reasonings.** The mathematical structures (15),(22),(19)and (24) act as a general paradigm for the derivation of models if the terms η, B, ε and G, that are called here ***interaction parameters***, are properly identified. Their identification needs to be referred to a detailed analysis of the interactions between subsystems. Suitable examples will be given in Section 6.

It is worth stressing that two different classes of models can be considered:

- ***Predictive models*** which describe the evolution of the system, given an initial condition and interaction parameters obtained by a suitable interpretation of empirical data.
- ***Explorative models*** that offer a panorama of different types of evolution according to an explorative guess on interaction parameters. These models can be used to select a strategy among several conceivable ones, while the investigation can be further refined in the framework of optimal control problems.

Finally, it is worth providing a critical analysis of the approach of this paper compared with the the existing literature based on agents methods [29], that consist in using the concept of agents' preference, and trying to maximize some objective function, depending on the case study: the objective function can be the welfare of the population, or the benefit of the dictator, or the benefit of some party or group of interest in a country. This maximization process can lead to some trade-off equations, which determine the final outcome. This general approach is developed tanks to game theory, both deterministic and stochastic. However, defining a preference function for all individuals in the same group cannot capture, in some circumstances, all the varieties and intensities of opinions within a large group of people. This is the reason why we want to describe this kind of phenomena using the probabilistic concept of distribution of preferences: by this approach it is possible to describe and explain complex social, economical, and political behaviors and phenomena from a more comprehensive point of view.

Indeed, according to many works of the recent literature such as [28], it is extremely unrealistic to think that individuals are perfectly rational, in the sense that all of them want to maximize some fixed utility functions: in reality, opinions and preferences can dynamically vary across time. This means that if we observe a socio-economical phenomena during a time interval $[0, T]$, the preference of a certain individual about the final outcome of this phenomena will be very likely to change from $t = 0$ to $t = T$. As an example, if we are dealing with a democratization problem, all individuals and parties who may play a significant role in this phenomenon, can radically change their preference and as a consequence their efforts to support or hinder the democratization process. This is just a single example among those introduced in the preceding sections. For this reasons, these socio-economical phenomena can clearly be described by an evolution equation of the distribution on the so-called socio-economical activity u. Instead of fixed preferences, it is fairly more realistic to deal with dynamic preferences. Moreover, what in the real world determines the final outcome of an individual opinion or strength to reach the final socio-economical outcome is not deterministic: it depends on the kind and the intensity of the encounters (interactions) among the different groups of individuals; after the encounter with a specific individual my position can be strengthened or, on the opposite, can be dramatically weakened, through a stochastic process. All the afore-mentioned models can be analysed with this new perspective, trying to understand which mechanisms rule the reaching of a final outcome, depending on the history of every single individual between $[0, T]$. This means that starting from a state u depending on the subsystem, the socio economical variable may change after probabilistic encounters.

This is the reason why, with our approach, even the understanding of the way the phenomenon takes place can be more precise, considering both the microscopic and the macroscopic point of view. Indeed, our models allow to observe the modification of the opinion of specific individuals, and the convergence towards the macroscopic event corresponding to the final equilibrium.

6. **Modelling and mathematical problems.** This section is devoted to the development of a modelling approach based on the mathematical frameworks proposed in Section 5 corresponding to closed and open systems. These are formally given by Eqs. (9) and (12), that have been particularized by Eqs. (15),(22),(19) and (24). The method is essentially based on a detailed modelling of the interaction terms, so that the mathematical structures can be fully characterized.

Therefore, it is worth focusing on which specific mathematical functions can suitably describe the encounter rate and the transition probability density functions. Moreover, the evolution of the system can be computed if the initial conditions $f_{i0}(u) = f_i(t = 0) \quad i = 1, \dots, n$, are given. These conditions may influence the afore-mentioned identification.

Let us first consider the ***encounter rate***. The identification refers to assessment of the symmetric matrix $[\eta_{ij}]$, whose entries η_{ij} generally depend on the state of the interacting pairs.

Referring to the practical examples described in the previous sections, it is reasonable to think that, the more two subsystems are close in their specific state variable u, the more frequently they interact, namely the encounter rate increases with decreasing distance.

Referring to the role of the characteristics of the interacting pair, it seems realistic to believe that political institutions interact with higher frequency if they are close in their tasks and duties: i.e. the dictator *entourage* interacts more with his ministers than with the people.

As a consequence, the encounter rate matrix entries $\eta_{ij} = \eta_{ji}$, can be made depending on the macroscopic quantities L_i, L_j, which identify the mean socio-economical state of the system. In particular, if we define L_{i0} and L_{j0} as the mean socio-economical state of the subsystems i and j respectively, at the initial time $t = 0$, the *distance in state*

$$d^0_{ij} =\mid L_{i0} - L_{j0} \mid ,$$

can be defined.

Therefore, the encounter rate between the subsystem i and j can be described by $\eta_{ij}(d^o_{ij})$, where in certain cases η_{ij} is an increasing function of d^o_{ij}, while in other cases is a decreasing function of d^0_{ij}. More in general, the encounter rate can be assumed to depend on the mean distance of the states at time t:

$$d_{ij}(t) =\mid L_i(t) - L_j(t) \mid \ . \tag{25}$$

Of course, if η_{ij} depends on time Eq.(15) needs to be properly rewritten to include this dependence. The simplest case is the one in which η_{ij} is simply a constant rate, not depending on any macroscopic quantity, in this case we indicate $\eta_{ij} = c_{ij}$, where c_{ij} is a suitable constant for the subsystems i and j.

The above reasoning focused on the interaction rate can be straightforwardly applied to open systems focused on the modelling of the rate μ_{ij} between the outer and inner system, treated as a constant or depending on the distance between the state of the interacting pairs.

Let us now consider the ***transition probability density functions*** B_{ij}. If the socio-variable u is defined over the whole real line $\mathbb{R}$, a reasonable assumption consists in defining B_{ij} as a *Gaussian density function*, $B_{ij}(m_{ij}(u_*, u^*), \sigma_{ij})$, where $[\sigma_{ij}]$ is a matrix, whose entries represent a dispersion factor assumed to be constant or depending on external factors, and $[m_{ij}(u_*, u^*)]$ is the matrix of the values around which the density is centered, namely the most probable value, depending on the two states of the interacting subsystems i and j.

The parameter m_{ij} can be modelled by taking into account the following law of change in state after the interaction between the subsystems i and j:

$$m_{ij} = u_* + \phi(u_*, u^*, \beta_{ij}), \tag{26}$$

where u_* is the state of the test subsystem and $\phi(u_*, u^*, \beta_{ij})$ its shift from the original value. $\phi(u_*, u^*, \beta_{ij})$ is a decreasing function in $\mid u_* \mid$, while β_{ij} is a parameter which determines the sign and the entity of this shift.

The function $\phi(u_*, u^*, \beta_{ij})$ can be assumed to depend on L_i and L_j since a change in state may depend on the mean value of the initial state of the subsystems. Moreover, also the case $\phi(u_*, u^*, \beta_{ij})$ depending on E_i and E_j can be studied, since a change in state can be due also to the energy values of the subsystems. In particular, if, as above, a distance in mean state is defined by $d_{ij} =\mid L_i - L_j \mid$, it can be assumed that $\phi = \phi(u_*, u^*, \beta_{ij}, d_{ij})$, meaning that it can be a decreasing or increasing function of such a distance.

Let us first introduce the mathematical entities which may characterize a competition for a secession. According to the definitions of the previous sections, the identification of the domain D_u for the socio-economical variable u is now necessary. If we deal with a country in which the Constitution and the law enforcement are well organized and respected, we define $D_u = [-1, 1]$: in nations where the democratic process is well established and it is not possible to overcome the limits settled by the laws, the variable u is not able to assume values external to the fixed interval $[-1, 1]$. In this case D_u corresponds to a bounded interval of the whole real line.

On the other hand, if we study a nation, where the juridical and political system is not powerful enough to guarantee the rule of law, we define $D_u = (-\infty, +\infty)$, since, in this case, the variable u can always reach extreme values. In this case D_u is no longer a bounded domain.

Remark 8. The structure reported in (6.2) does not hold in full generality. However, it can be justified considering that in every ordinary interaction, individuals generally agree, approaching their state u, or strongly disagree, getting more distant in their state u, after a common discussion.

Remark 9. Conservative interactions between i and j are such that $L_i + L_j = c_L$, where c_L is a constant. Interactions can also be conservative in energy if $E_i + E_j = c_E$, where again c_E is a constant. Conservation holds when the sum of mean states and energies do not vary before and after the interactions, although each mean state and energy can vary. This means that each subsystem can change moments, but the sum of the moments remains the same, during all the evolution. If the conservativeness of the interactions is assumed, the terms α_{ij} must be such to satisfy the constraint given by the conservative constraint equation for L_i or E_i.

7. **Modelling the competition for a secession.** This section develops the approach proposed in the preceding two sections to the modelling of competition for a secession in the case of a closed system. Specifically, the following topics are dealt with in the four following subsections:

i) Characterization of the model of competition for a secession;

ii) Sample simulations focused to visualize the role of the parameters of the model on the behavior of the solutions;

iii) A critical analysis to develop and improve the model.

7.1. **Characterization of the model.** Let us now consider the modelling of a competition for a secession, referred to a system corresponding, in this specific problem, to a nation isolated from the external environment.

Let us assume that the nation is divided into a richer and a poorer region, so that the richer part hopes to benefit from a secession in terms of national income and taxes. Many cases of nowadays reality in which this is happening, or has already happened, can be figured. Different oriented political parties and interest groups interact in the global nation in order to support the secession or not. The specific model proposed in what follows is characterized by a small number of variables and very simple interaction functions. The aim is to show that, even in a simple case, the model can describe some interesting phenomena. The last subsection critically analyses the model, proposing some guidelines to improve it.

Following the approach proposed in the preceding sections, the first step of the modelling consists in identifying the functional subsystems that compose the overall system. Accordingly, let us suppose to decompose the system into the following subsystems:

- Political parties,
- Unions,
- White collars.

Let us just focus on the above three subsystems to define a simple model suitable to describe the evolution of the system using a small number of parameters. Bearing in mind that each subsystems can be anyway represented in a much more complex way considering the different orientation (left, right and center) that are present in political parties, unions and white collars. We define the following probability densities:

- $f_1(t,u)$, describes the distribution over the state of the subsystem ***political parties***,
- $f_2(t,u)$, describes the distribution over the state of the subsystem ***unions***,

• $f_3(t,u)$, describes the distribution over the state of the subsystem ***white collars***.

The second step consists in modelling the interaction terms, namely the matrix of the encounter rates η_{ij} and the parameters of the transition probability density. Let us deal with the modelling of the encounter rate η_{ij} according to the following assumption:

Hp. 1. *$\eta_{ij} = \eta_{ji}$ is a constant*

$$\eta_{ij} = \eta_{0ij}. \tag{27}$$

In particular we suppose that η_{ij} attains the largest value when the interactions take place within the same subsystems:

$$\eta_{11} = \eta_{22} = \eta_{33} = \eta_0 = 1\,,$$

while

$$\eta_{12} = \eta_{21} = \alpha_1 < \eta_0, \qquad \eta_{13} = \eta_{31} = \alpha_2 < \alpha_1\eta_0, \qquad \eta_{23} = \eta_{32} = \alpha_3\eta_0\,,$$

where $\alpha_2 < \alpha_3 < \alpha_1$.

The values of η_{ij} are summarized in the following matrix:

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ \alpha_1 & 1 & \alpha_3 \\ \alpha_2 & \alpha_3 & 1 \end{pmatrix}. \tag{28}$$

Remark 10. The entries of the above matrix are simply characterized by three parameters.

Let us now model the transition probability density function. The modelling is developed according to the following assumption consistent with the approach proposed in subsection (6.2).

Hp. 2. *$B_{ij}(m_{ij}(u_*,u^*),\sigma_{ij})$ is a Gaussian density function such that:*
i) σ_{ij} is a constant for each interacting pairs;
ii) the mean value is defined as follows as

$$m_{ij} = u_* + \beta_{ij}(u_* - u^*), \tag{29}$$

where β_{ij} is a small parameter which can assume either positive or negative value.

Remark 11. If the dispersion factor σ_{ij} tends to zero, the transition probability density function is as follows:

$$B_{ij}(u_*,u^*;u) = \delta(u - m_{ij}(u_*,u^*))\,, \tag{30}$$

where δ is the Dirac distribution function.

Moreover, we suppose that the interaction within the same subsystem does not change the state of the subsystem:

$$\beta_{11} = \beta_{22} = \beta_{33} = 0\,,$$

and that the probability distribution is symmetric:

$$\beta_{12} = \beta_{21} = \beta_1\,, \quad \beta_{13} = \beta_{31} = \beta_2\,, \quad \beta_{23} = \beta_{32} = \beta_3\,.$$

Instead β_1,β_2, and β_3 are positive if we suppose to deal with a *dialectic model* and negative if we are dealing with a *non dialectic model*. The values of β_{ij} are reported in the following matrix:

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 \\ \beta_1 & 0 & \beta_3 \\ \beta_2 & \beta_3 & 0 \end{pmatrix}. \tag{31}$$

In the first case, namely dialectic model, we deal with a utopian reality in which parties always find a compromise solution, getting nearer in their opinions. In the second case, namely non dialectic model, we deal with a non realistic world in which every party strengthens and radicalizes its position after encounters.

This is the reason why, we propose a third model in which the parameters are the same as before, except for an extra parameter d_c, which represents a critical distance such that:

i) if $| u_* - u^* | < d_c \Rightarrow \beta_1, \beta_1, \beta_3 > 0$;

ii)if $| u_* - u^* | \geq d_c \Rightarrow \beta_1, \beta_1, \beta_3 < 0$.

This means that the model can become either dialectic or non dialectic depending on the distance in state of the interacting subsystems: if d_c is big enough the parties can reach an agreement while if d_c is not, there is a conflict solution. Notice that it is possible to define more than a single critical distance d_c, so that there are many intervals in the real line in which the model is dialectic and others in which it is not.

Taking into account Hp.1 and Hp.2, it is possible to write a detailed equation for the model for every density function f_i, $i = 1, 2, 3$ in the case $B_{ij}(u_*, u^*; u) = \delta(u - u_* + \beta_{ij}(u_* - u^*))$:

$$\left\{\begin{array}{rcl} \partial_t f_1(t,u) & = & \dfrac{\alpha_1}{| 1-\beta_1 |}\displaystyle\int_{\mathbb{R}} f_1(t, \frac{u-\beta_1 u^*}{1-\beta_1}) f_2(t,u^*) du^* \\ & + & \dfrac{\alpha_2}{| 1-\beta_2 |}\displaystyle\int_{\mathbb{R}} f_1(t, \frac{u-\beta_2 u^*}{1-\beta_2}) f_3(t,u^*) du^* - (\alpha_1+\alpha_2) f_1(t,u) \\ \partial_t f_2(t,u) & = & \dfrac{\alpha_1}{| 1-\beta_1 |}\displaystyle\int_{\mathbb{R}} f_2(t, \frac{u-\beta_1 u^*}{1-\beta_1}) f_1(t,u^*) du^* \\ & + & \dfrac{\alpha_3}{| 1-\beta_3 |}\displaystyle\int_{\mathbb{R}} f_2(t, \frac{u-\beta_3 u^*}{1-\beta_3}) f_3(t,u^*) du^* - (\alpha_1+\alpha_3) f_2(t,u) \\ \partial_t f_3(t,u) & = & \dfrac{\alpha_2}{| 1-\beta_2 |}\displaystyle\int_{\mathbb{R}} f_3(t, \frac{u-\beta_2 u^*}{1-\beta_2}) f_1(t,u^*) du^* \\ & + & \dfrac{\alpha_3}{| 1-\beta_3 |}\displaystyle\int_{\mathbb{R}} f_3(t, \frac{u-\beta_3 u^*}{1-\beta_3}) f_2(t,u^*) du^* - (\alpha_2+\alpha_3) f_3(t,u) \end{array}\right. \tag{32}$$

An analogous model can be derived in the case of discrete socio-variable $u \in [-1, 1]$. In this case, f_1, f_2, f_3 are discrete density functions for each subsystem Union, Political parties and White collars. The domain $[-1, 1]$ is defined, in this specific case, as follows

$$I_u = \{u_1 = -1\,,\, u_2 = -0.5\,,\, u_3 = 0\,,\, u_4 = 0.5\,,\, u_5 = 1\} \tag{33}$$

representing $H = 5$ different level of opinions. The encounter rate matrix is the same as before, while in a discrete model we need a different definition of the discrete probability density function $B_{ij}^{pq}(h) = B_{ij}(u_p, u_q; u_h)$, representing the probability density that a particle with state u_p of the population i falls in the state u_h, after the encounter with a particle with state u_q of the population j. We look for a discrete function such that

$$\forall i, j, \quad \forall p, q \,: \qquad \sum_{h=1}^{5} B_{ij}^{pq}(h) = 1. \tag{34}$$

Let us introduce, as before, a critical distance d_c. Following the continuous model, we can define the discrete transition probability density function as follows. If $p = q \Rightarrow B_{ij}^{pq}(h = p) = 1, \quad B_{ij}^{pq}(h \neq p) = 0$. If $p \neq q$, $| p - q | \leq d_c$, the model is dialectic, namely, if $p < q$

$$B_{ij}^{pq}(h = p + 1) = \beta, \quad B_{ij}^{pq}(h = p) = 1 - \beta, \quad B_{ij}^{pq}(h \neq p, p + 1) = 0;$$

if $p > q$

$$B_{ij}^{pq}(h = p - 1) = \beta, \quad B_{ij}^{pq}(h = p) = 1 - \beta, \quad B_{ij}^{pq}(h \neq p, p - 1) = 0.$$

If $p \neq q$, $\mid p - q \mid \geq d_c$, the model is non dialectic, namely, if $p < q$

$$B_{ij}^{pq}(h = p - 1) = \beta, \quad B_{ij}^{pq}(h = p) = 1 - \beta, \quad B_{ij}^{pq}(h \neq p, p - 1) = 0;$$

if $p > q$

$$B_{ij}^{pq}(h = p + 1) = \beta, \quad B_{ij}^{pq}(h = p) = 1 - \beta, \quad B_{ij}^{pq}(h \neq p, p + 1) = 0,$$

where $\beta > 1/2$.

Taking into account the previous definitions for η_{ij} and B_{ij}^{pq}, it is possible to obtain the following 3×5 systems of ordinary differential equations:

$$\frac{df_i^h}{dt} = \sum_{j=1}^{3} \left(\sum_{p=1}^{5} \sum_{q=1}^{5} \eta_{ij} B_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=1}^{5} \eta_{ij} f_j^q \right), \tag{35}$$

where $i = 1, 2, 3$ and $h = 1, 2, 3, 4, 5$.

7.2. **Simulations of a model of political conflicts.** Simulation are obtained by solving the initial value problems generated by model (32) or (35) linked to suitable initial conditions. The two problems can be formally written as follows:

$$\begin{cases} \partial_t f_i(t, u) = J_i[f](t, u), \\ \\ f_{i0}(u) = f_i(t = 0, u) \end{cases} \tag{36}$$

for $i = 1, \dots, 3$ and

$$\begin{cases} \frac{df_1}{dt}(t, u) = J_i^h[f](t), \\ \\ f_{i0}^h = f_i^h(t = 0) \end{cases} \tag{37}$$

for $i = 1, \dots, 3$; $h = 1, \dots, 5$, and where the right-hand side terms correspond, respectively, to Eqs.(32) and (35).

A qualitative analysis of problem (36), for a class of equations with the same properties has been proved in paper [1] where it has been proven, by application of classical fixed point theorems, existence, uniqueness, and positivity of the solutions in the space of the functions integrable over the whole real line. Regularity of the solution can be proved under suitable assumptions on the smoothness of the initial data.

The qualitative analysis of Problem (37) needs a technically different approach. The existence proof is given in the paper by Bertotti and Delitala [13], who have been able also to show several properties on the asymptotic behavior of the solutions [14], both for closed and open systems, when the number of microscopic states is small. On the other hand, only partial results of the analysis are available, that makes the contribution of simulations as essential to depict the configurations reached asymptotically in time.

In particular, referring to the model described in the previous section, simulations are developed to obtain the time evolution of the three subsystems under consideration, respectively political parties, unions and white collars. The following parameters representing the encounter rate and the transition probability density: $\alpha_1 = 0.75$, $\alpha_3 = 0.45$, $\alpha_2 = 0.25$, and $\beta = 1$, are adopted, while the interval of opinions, spanning in the interval $[0, 1]$, is discretised into 5 different levels, representing the 5 different opinions which can be assumed within each subsystems.

Keeping all these parameters fixed, the goal of the simulations is to investigate the role of the critical distance d_c in the evolution of f_1, f_2 and f_3, given random uniformly distributed initial conditions. Simulations are developed in two distinct cases: the case in which the two extreme opinions (namely opinion 1 and opinion 5 representing the extreme attitude towards or against secession) cannot vary, although encountering people with other opinions; and the case in which the two extreme opinions are free to move between opinion 0 and opinion 1. All the figures reported hereby are such that continuous line corresponds to opinion 1, dashed line to opinion 2, thick line to opinion 3, dot-dashed line to opinion 4, and very thick line to opinion 5.

In the case with fixed extreme opinions, all subsystems have the same kind of behavior and the parameter d_c does not play a significant role in the evolution of f_1, f_2, f_3: for every value of $d_c = 0, 1, 2, 3, 4$ the behavior is the one shown in figures 1, 2 and 3. The two extreme opinions are the unique to survive for every d_c, no matter which subsystem examined: at the equilibrium all the individuals in each subsystems are split into two groups having different extreme ideas. The reason of this behavior is the following:

If extreme opinions are fixed, they play the role of attractor opinions, namely, once an individual reaches opinion 1 or 5, it cannot move further.

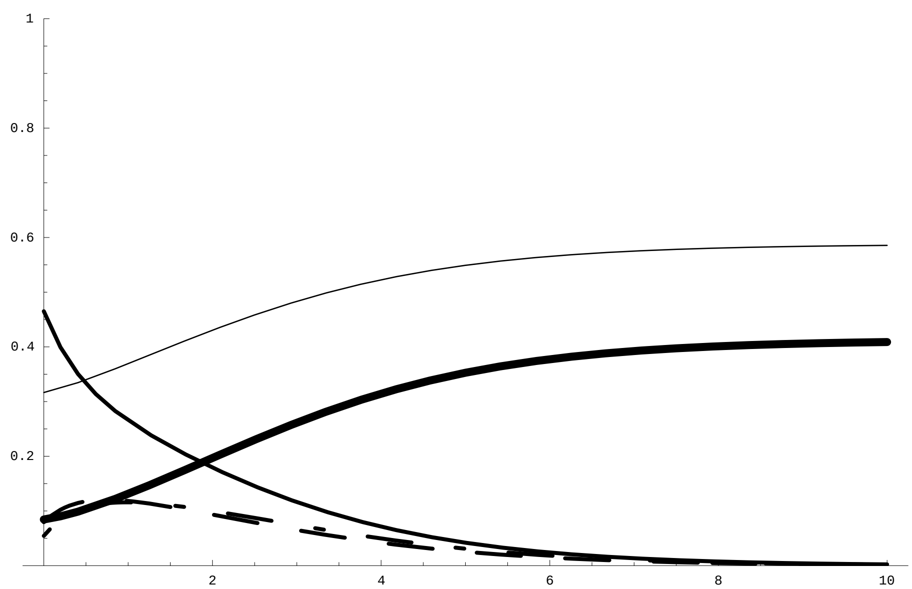

FIGURE 1. Evolution of the components of f_1 in the case of fixed extreme opinions

On the other hand, if extreme opinions are allowed to vary, the evolution scenario is completely different and the critical distance d_c does assume a significant role. When $d_c = 0$ or $d_c = 1$, namely the cases in which individuals tend to repulse different opinions, the evolution of f_1, f_2, f_3 is very similar to the one above, therefore figures are omitted.

When $d_c = 2$, the evolution begins to change: both a solution where opinion 1 and 5 survive as before is found, and a dialectic solutions appears, where the central opinion (opinion 3) is generally the dominant at equilibrium, sometimes with a second surviving opinion (opinion number 2 or 4). Notice that with $d_c = 2$ the convergence is much slower than that in the previous simulations. See figures 4, 5, 6, 7, 8, 9, 10, 11, and 12.

When $d_c = 3$, convergence is fast and the central opinion always survives, sometime jointly with opinions 2 or 4, while extreme opinions 1 and 5 never survive, as shown in figures 13, 14, 15, 16, 17, and 18.

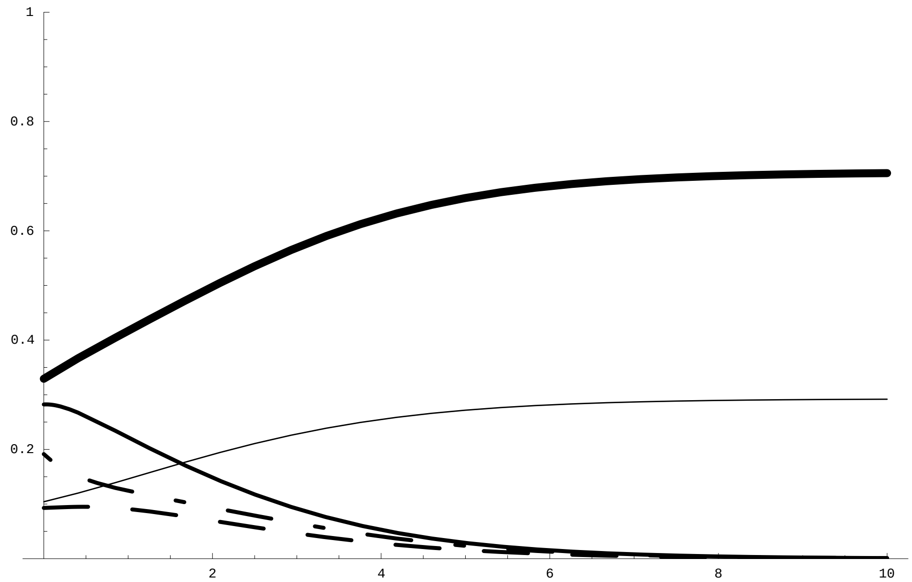

FIGURE 2. Evolution of the components of f_2 in the case of fixed extreme opinions

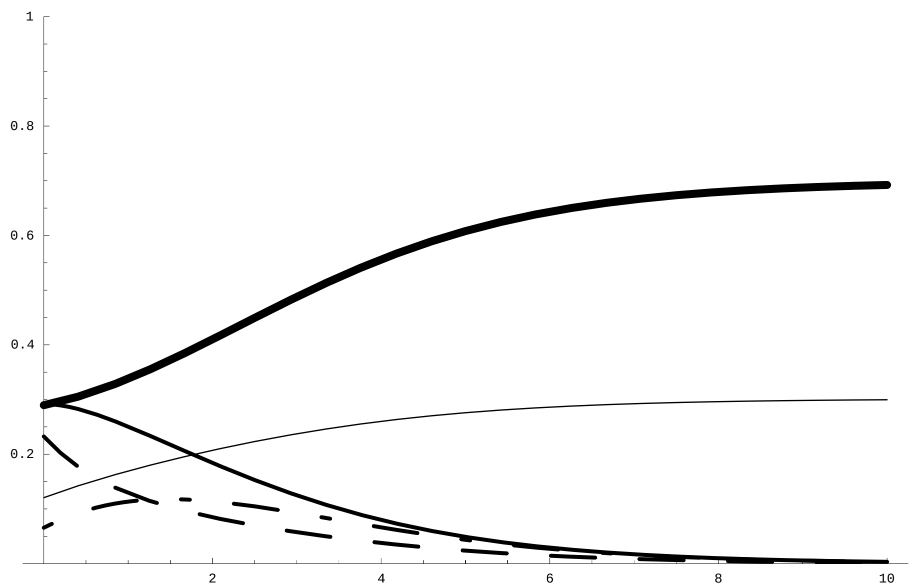

FIGURE 3. Evolution of the components of f_3 in the case of fixed extreme opinions

Finally, for $d_c = 4$, the opinion that survives is again the central one, generally alone, as shown in figures 13, 14,15.

In conclusion, as summarized in Table 1 (notice that i.c. means *initial condition*), when extreme opinions do not move, i.e. extremists remain extremists with no possibility to change, even if interacting with moderates, extreme opinions are the unique to survive.

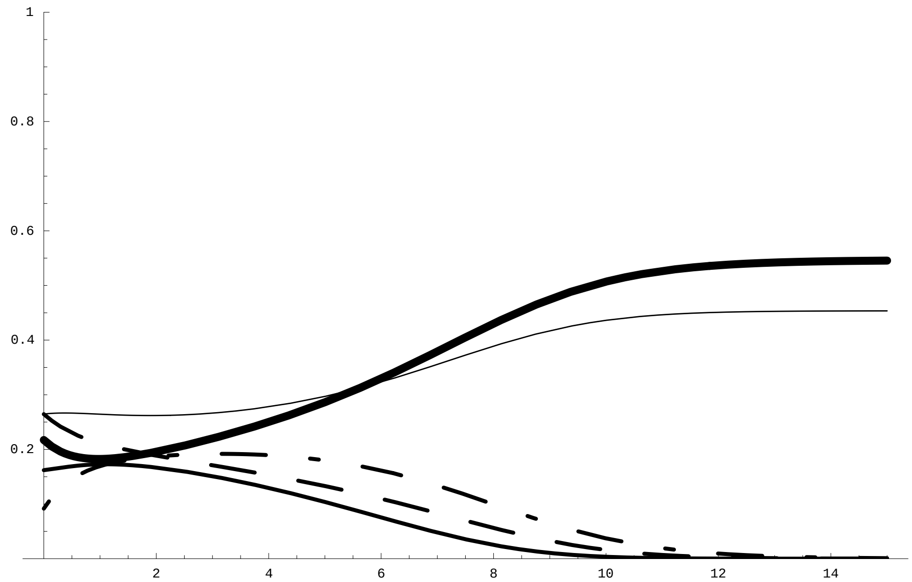

FIGURE 4. Evolution of the components of f_1, $d_c = 2$ extreme opinions survive

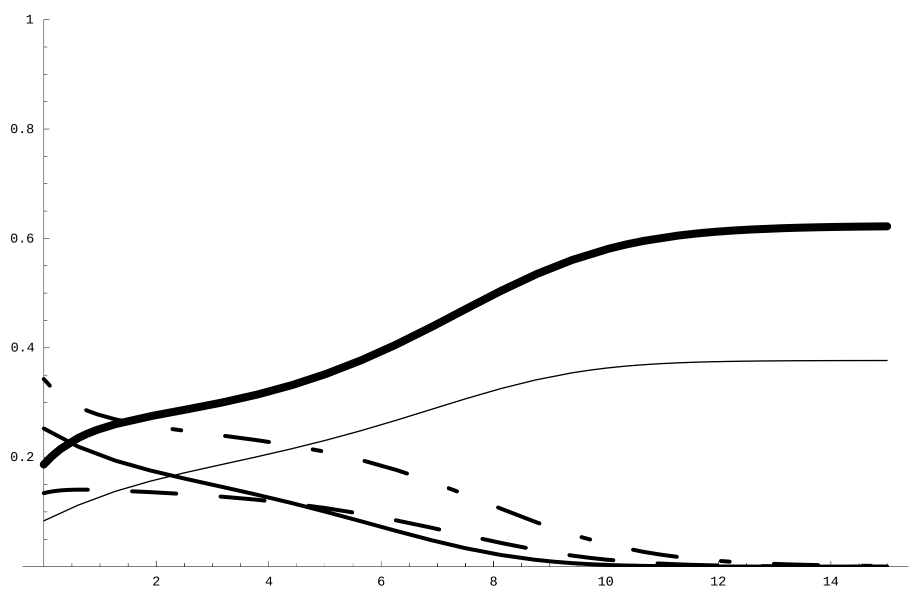

FIGURE 5. Evolution of the components of f_2, $d_c = 2$ extreme opinions survive

There are no significant differences with different d_cs: if radical people does not change their mind d_c has no role no matter how moderate other people are, sooner or later moderates will be attracted and affected by extremists. There is no possibility of a compromising solution.

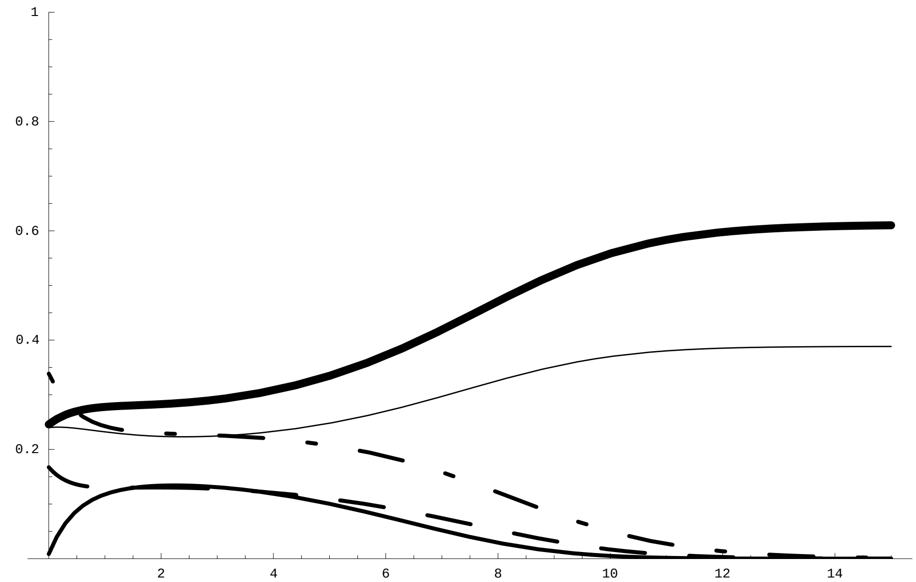

FIGURE 6. Evolution of the components of f_3, $d_c = 2$ extreme opinions survive

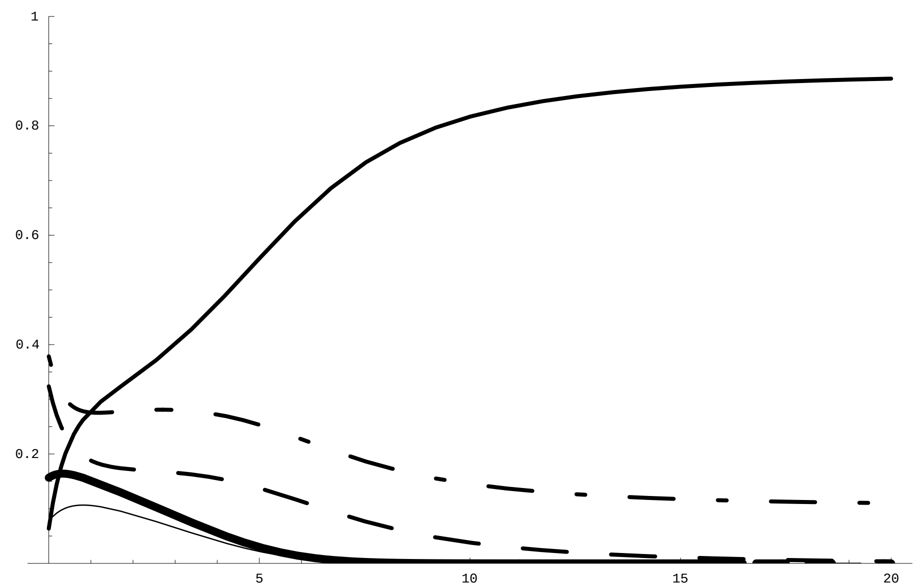

FIGURE 7. Evolution of the components of f_1, $d_c = 2$ central opinion survives

When extremists can move towards moderate opinions, as summarized in Table 2, d_c has a fundamental role. For low values of d_c results are very similar to the case of fixed extreme opinions, since people prefer to be repulsed than to be attracted by someone else's opinions. When $d_c = 2$, the transition takes place: compromise solutions begin to appear. It is not surprising that in this case convergence is slower: this is the minimum distance for which central opinion can survive and it takes longer to reach an equilibrium.

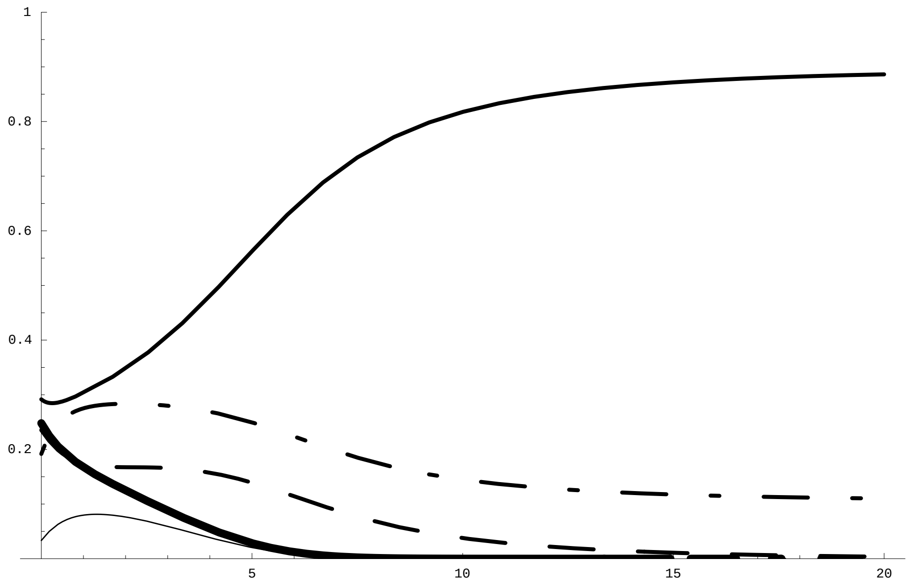

FIGURE 8. Evolution of the components of f_2, $d_c = 2$ central opinion survives

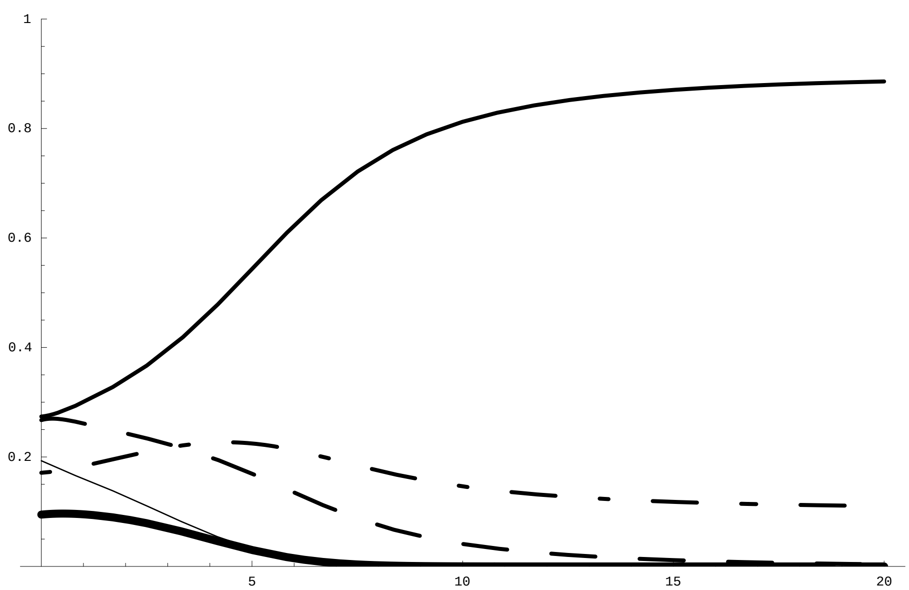

FIGURE 9. Evolution of the components of f_3, $d_c = 2$ central opinion survives

When $d_c = 3$, and especially $d_c = 4$, convergence is faster and central opinions always survive while extreme ones do not. If d_c is big, people are no longer repulsed but attracted by someone else's ideas and everyone goes towards the central moderate opinion, so that finally all the population agrees!

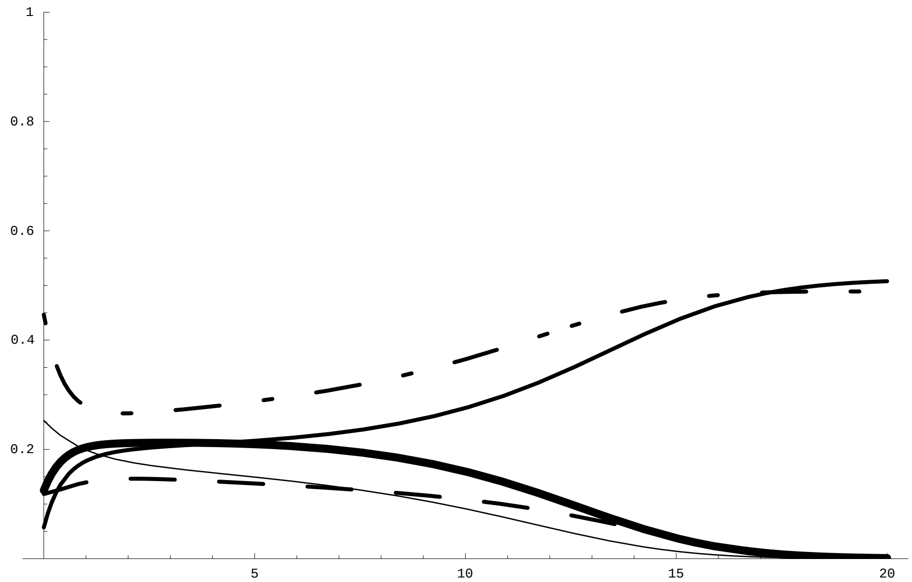

FIGURE 10. Evolution of the components of f_1, $d_c = 2$ two central opinions survive

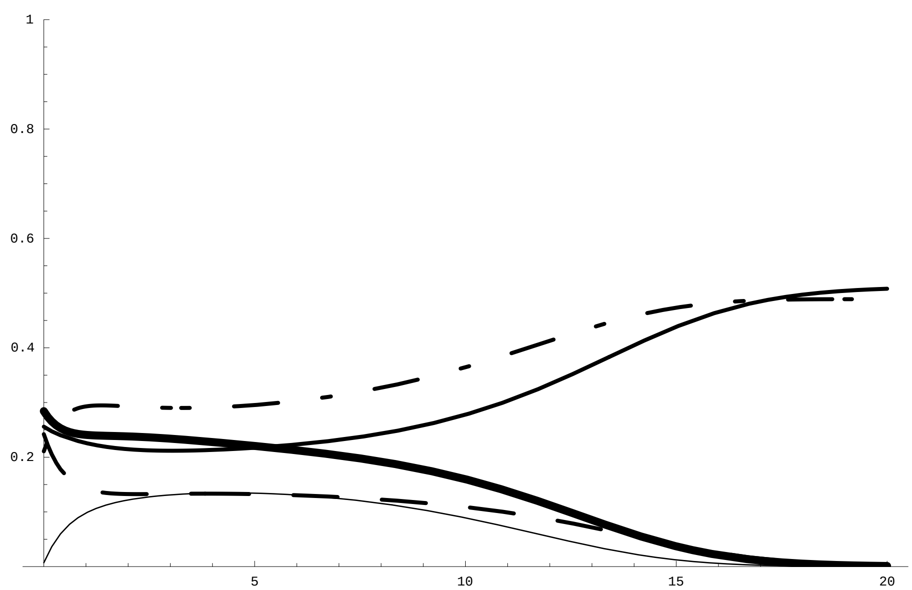

FIGURE 11. Evolution of the components of f_2, $d_c = 2$ two central opinions survive

7.3. **Reasonings on further modelling issues.** The mathematical model proposed and analysed in the preceding subsection has to be regarded as a simple application to show the theoretical approach of this paper applied to a specific case selected among those introduced in Section 2. Indeed, it is a toy model designed with explorative aims.

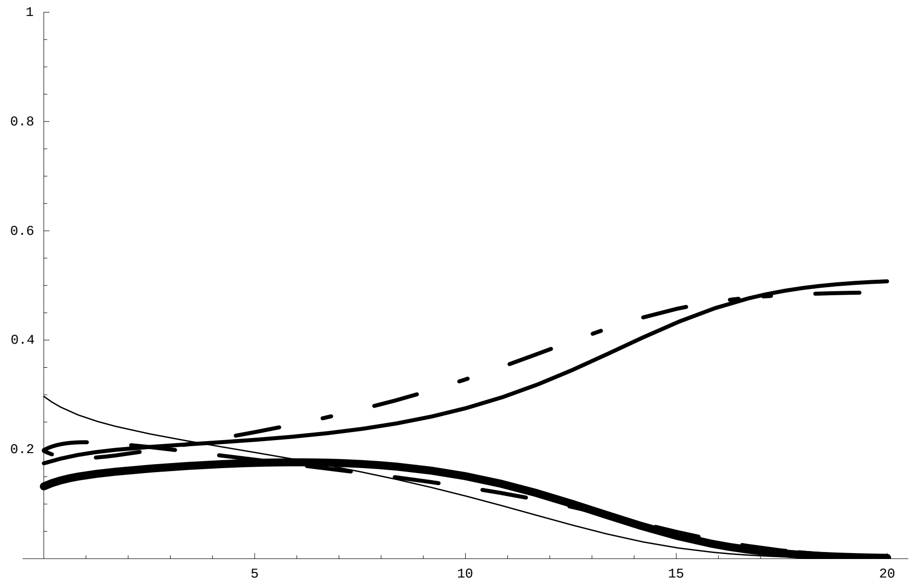

FIGURE 12. Evolution of the components of f_3, $d_c = 2$ two central opinions survive

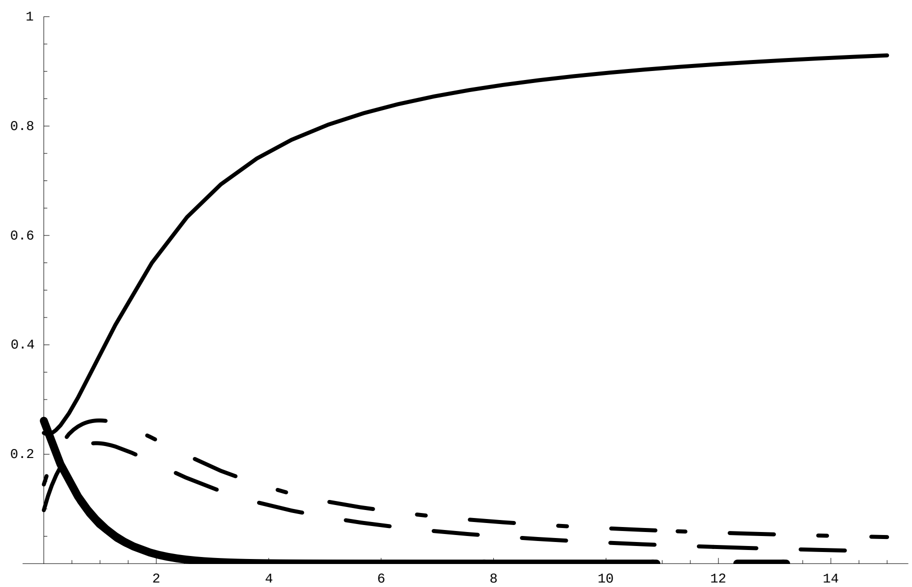

FIGURE 13. Evolution of the components of f_1, $d_c = 3, 4$ one central opinion survives

Some interesting aspects have been shown focused on the role of persuasion due to exchange of opinions related also to the presence of extremist positions. As we have seen, fixed extreme opinions play the role of attractor of opinions, that makes useless the exchange of opinions. This phenomenon is currently observed in politics where conflicts are often artificially radicalized to avoid a democratic dialogue, possibly based on ideas

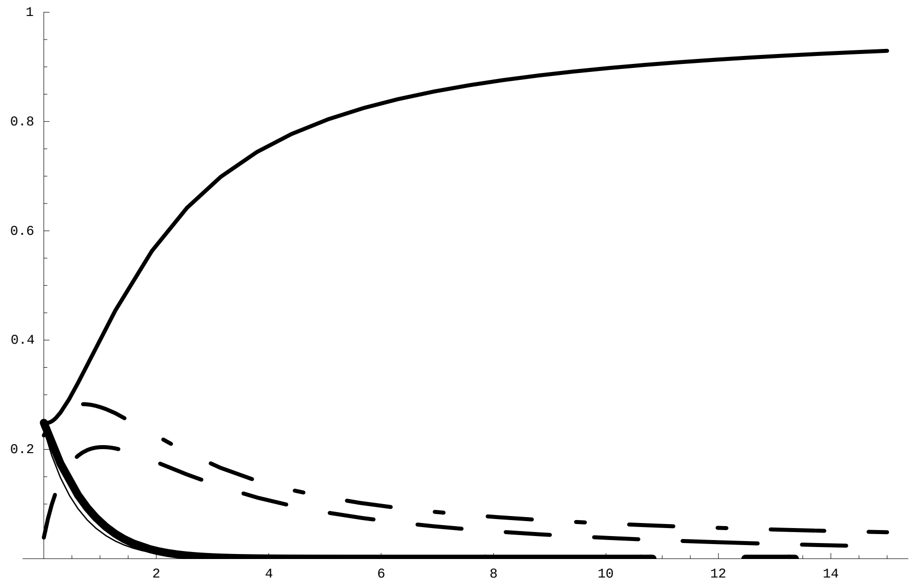

FIGURE 14. Evolution of the components of f_2, $d_c = 3, 4$ one central opinion survives

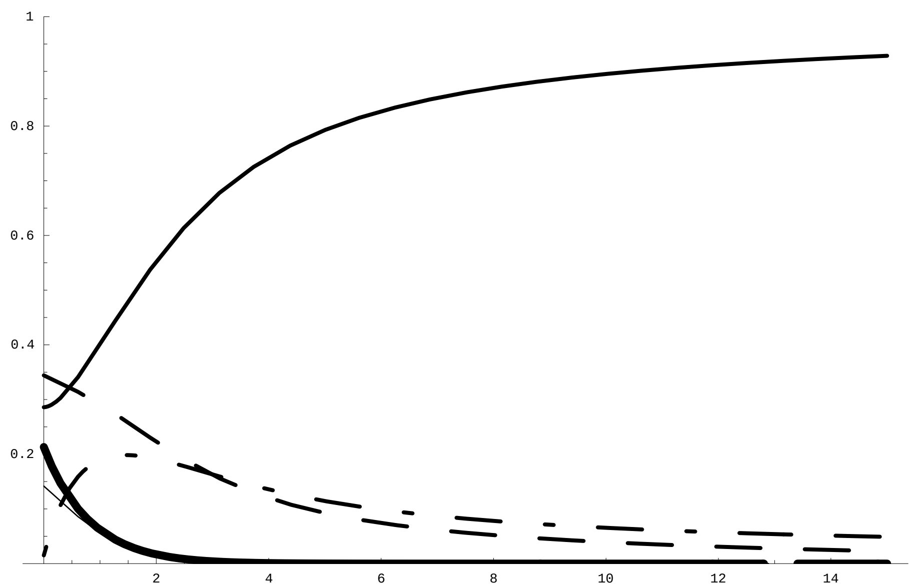

FIGURE 15. Evolution of the components of f_3, $d_c = 3, 4$ one central opinion survives

and social projects, that may eventually shift the majority of holders of a certain political idea towards an undesired direction.

The modelling has been developed in the case of absence of uniformly external actions. Specifically, an interesting modelling perspective refers to a detailed analysis of the role of external actions. The dynamics that has been shown in the simulations suggest to avoid

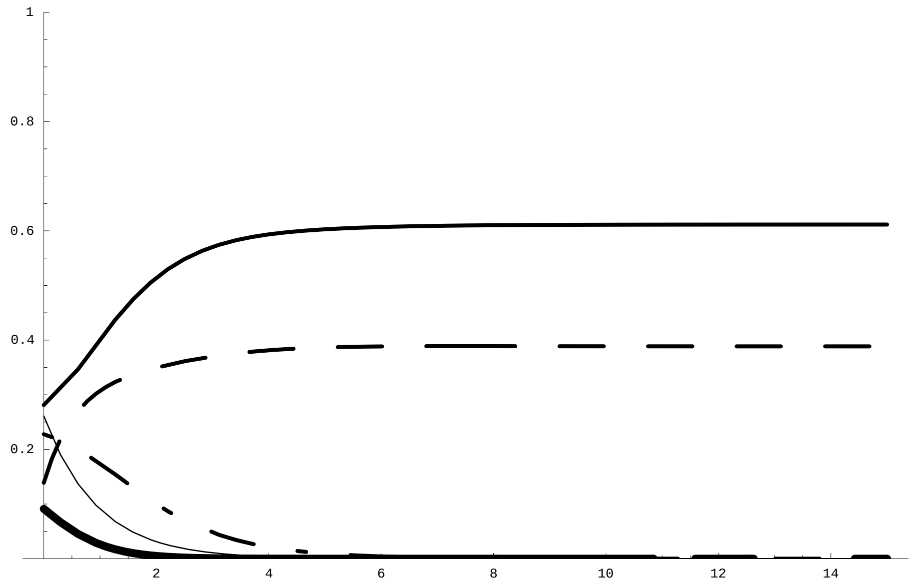

FIGURE 16. Evolution of f_1, $d_c = 3, 4$ two central opinions survive

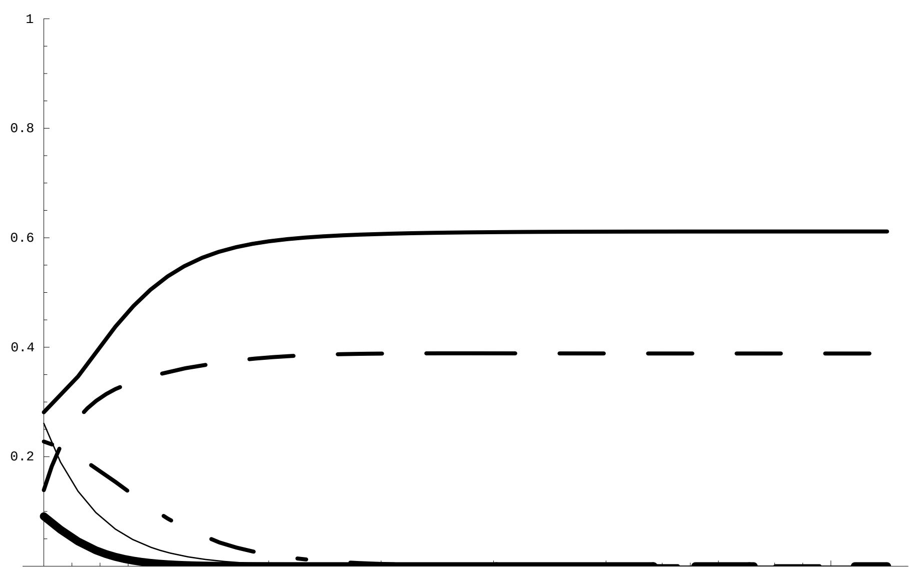

FIGURE 17. Evolution of f_2, $d_c = 3, 4$ two central opinions survive

external uniformly applied to the whole range of opinions. Rather, the actions should be concentrate on a few, may even be one, specific opinion. The investigation should also consider the cost of persuasive actions and the optimization of the whole persuasive action at fixed costs.

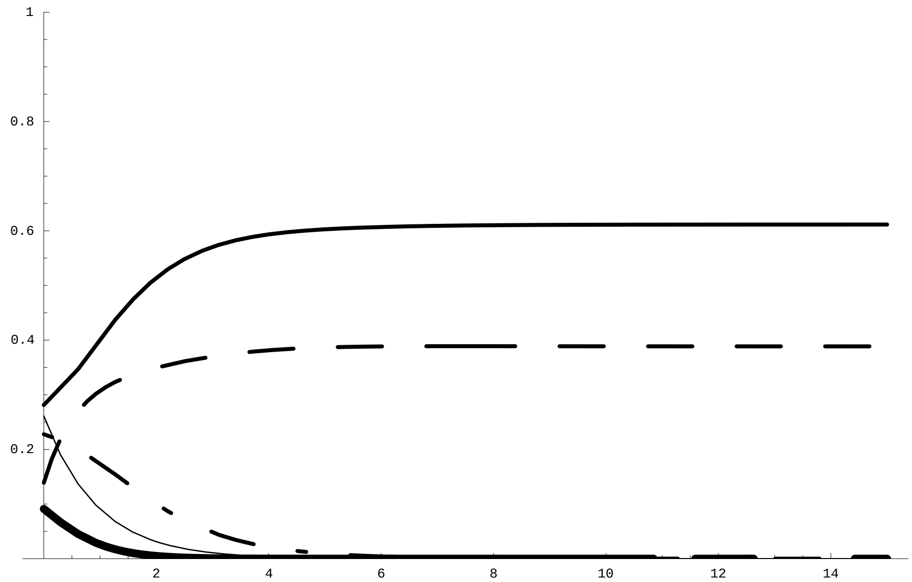

FIGURE 18. Evolution of f_3, $d_c = 3, 4$ two central opinions survive

Fixed opinions 1 and 5			
	opinions 1 and 5	opinions 2 and 4	opinion 3
$d_c = 0$	always	never	never
$d_c = 1$	always	never	never
$d_c = 2$	always	never	never
$d_c = 3$	always	never	never
$d_c = 4$	always	never	never

Summarizing Table 1 - Survivance of opinions

Mobile opinions			
	opinions 1 and 5	opinions 2 and 4	opinion 3
$d_c = 0$	always	never	never
$d_c = 1$	always	never	never
$d_c = 2$	depending on i.c.	depending on i.c.	depending on i.c.
$d_c = 3$	never	depending on i.c.	always
$d_c = 4$	never	depending on i.c.	always

Summarizing Table 2 - Survivance of opinions

8. **Critical analysis and perspectives.** A mathematical approach to social and behavioral economy has been developed in this paper based on the methods of the kinetic theory for active particles and on the characterization of functional subsystems to identify the socio-economical entities that interact in the environment under consideration. Interactions are modelled by stochastic games. An application to modelling the competition for a secession is also dealt with, although at a very preliminary stage with the aim of testing the potential ability of the mathematical approach for the applications and, specifically, to model phenomena of interest to the complex class of systems under consideration.

Considering that this paper aims at initiating a research program and is focused on methodological issues, leaving further applications to future activity, this final section proposes some further speculations related to a deeper understanding of the concept of functional subsystems and to a conceivable development of the methodological approach. Specifically, the three subsections, that conclude this paper deal with the following topics:

i) Decomposition of the whole system into functional subsystems with reference to scaling problems;

ii) Aggregation of functional subsystems into a new subsystem with greater size, or fragmentation of a functional subsystem into two subsystems with smaller size;

iii) Reasonings about the interaction of functional subsystems interconnected in networks.

The analysis is technically referred to the three specific classes of models proposed in the preceding sections. However, it remains at a preliminary level leaving a deeper insight to appropriate research programs. The contents of the following subsection refer to the already cited papers by Bertotti and Delitala [12], [13], [14], that offer a valuable source of ideas for the analysis of this paper. A closure on general issues is given in the last subsection.

8.1. **Additional reasoning on the concept of functional sub-systems.** The modelling approach proposed in this paper is based on the concept of functional subsystem considered as an aggregation of groups of interest expressing a common socio-economical function. It is worth stressing that this concept is flexible being related to specific events and issues that are object of modelling. Therefore, if the general context changes, the characterization and size of the functional subsystems also may have to be modified.

In principles, the function expressed by a subsystem can be a vector; however, the complexity of modelling interactions suggests to refine the identification of the subsystem by an additional decomposition to obtain that each of them expresses one function only. The examples reported in Section 2 clarify the above concept considering that, in each example, the identification of the groups of interest (identified as functional subsystems and reported in the first line of the three tables) refers to the general context (reported in the second line), while the function that is expressed is reported in the third line. When the general context and the socio-economical analysis changes, the identification of the groups of interest and their expression has to be modified.

8.2. **Aggregation and fragmentation of functional sub-systems.** The modelling approach is based on the assumption that the number of functional subsystems is constant in time. On the other hand, their aggregation or fragmentation can possibly occur in some circumstances. For instance the afore mentioned events may occur referred to the size of the subsystems or to the presence of radicalized opinions or interests.

The research perspective of including these events is definitely interesting. However, it cannot be pursued by a straightforward generalization of the approach proposed in this paper, while a further development of the mathematical structure developed in Section 5 cannot describe aggregation or fragmentation events unless properly modified. Some constructive suggestions are offered in Chapter 4 of Ref. [7], that may be possibly adapted to the class of systems under consideration.

8.3. **Interactions in networks.** The modelling dealt with in the preceding section considers interactions that do not depend on the geometry of the system and specifically on the localization of the interacting subsystems. This simplification is valid, when communications occur through delocalized devices: for instance media, internet, and so on. On the other hand, communications may be constrained by networks that organize and select the dialogue between pairs of functional subsystems.

The above research perspective needs, as in the previous case, an additional development of the mathematical framework that may possibly include multiple interactions [27], [8].

8.4. **Closure.** The contents of this paper has been focused on the modelling of complex socio-economical systems where individual behaviors play a crucial role in the interactions among functional subsystems. Interactions are modelled by stochastic games, that define the output of the interactions when the input states are given.

Interactions are stochastic as deterministic rules are not followed. Personal behaviors generate not only deviations from the most probable inputs, but may possibly change due to environmental conditions. For instance the onset of panic conditions.

The mathematical frameworks derived in Section 5 aim at offering the necessary background towards the derivation of specific models such as that of Section 6. Actually, this model can be regarded as a simple application proposed to show the application of the method. Although simulations have already shown the ability to describe interesting phenomena, further developments are necessary to enrich its descriptive ability. Indeed, some perspective ideas have already been considered, some of them induce further conceivable development of the mathematical approach.

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On the modelling and simulation of the competition for a secession under media influence by active particles methods and functional subsystems decomposition

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ABSTRACT

This paper deals with the development of a mathematical model for complex socio-economical systems, where external actions play a key role. The aim of the paper is to show the emergence of collective behaviors or macroscopic trends from individual based interactions, where agents are identified by functional subsystems. The approach is based on the methods of the mathematical kinetic theory for active particles, which describes the evolution of large systems of interacting entities which are carriers of specific functions: in our specific application socio-political activities subjected to the influence of the media.

1. Introduction

A new mathematical approach to modelling complex social-economical systems has been recently proposed in [1] as a natural development of the mathematical kinetic theory for active particles [2], already applied in various fields of life sciences, e.g. to model multicellular systems [3], and social behaviors of interacting individuals [4,5]. This mathematical theory describes the evolution of the probability distribution over the microscopic state, called activity, of several interacting entities called active particles. The equation which models the evolution is derived by a conservation balance in the elementary volume of the space of the microscopic states, where the inlet and outlet flows are determined by interactions among active particles.

The above mathematical approach has been developed in [1] to model complex socio-economical systems, where individual behaviors and interactions may play a significant role on the evolution of the system. Indeed, living systems have the ability to think, and as a consequence, to react to external actions, without following rules constant over time. A basic reference is the so called *behavioral economics*, where deterministic rules may be stochastically perturbed by individuals behaviors, that not only show random fluctuations, but may be substantially modified by external environments, e.g. depressive or panic situations. Many attempts to explicate these phenomena can be found in [6–14], where different models of opinion dynamics are described. Other references with a more socio-psychological approach can be found in [15, 16]. Moreover, the literature of the last five years, with works as [17–20], has developed opinion dynamics models, focused to understand the role of extremists in the evolution of the system. These ideas were preliminarily presented in [21].

These classes of models are finalized to show the emergence of collective behaviors and macroscopic trends, as a consequence of individual based interactions. In our approach, the entities which interact to determine the emerging equilibria are ***functional subsystems***: according to the mathematical theory developed in [1], the overall system is decomposed into

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Table 2.1
Competition for a secession under influence of media

Global nation				
Region1		Region2		External action
Political parties	Interest groups	Political parties	Interest groups	Media
$u > 0$ – pro	$u < 0$ – against	$u > 0$ – pro	$u < 0$ – against	Action over u

functional subsystems that refer both to the functions expressed by the socio-economical system, which is the system under consideration, and to the observation and representation scale used in the mathematical modelling process. Therefore each sub-system is related to a specific socio-economical function, called ***activity***. The time-evolution of the whole system is modelled by mathematical differential equations that describe the evolution of the probability distribution over the activity variable. This general mathematical framework is specifically modified in this present paper to model the phenomenon of competition for a secession under the action of media. The analysis is focused on the critical analysis of different strategies to obtain, from the competition, the desired output. Suitable simulations contribute to this investigation.

In detail, the contents of the paper are organized in five sections, which follow the above introduction.

Section 2 deals with a description of the complex socio-economical system under consideration: the competition for a secession under the influence of external actions. Moreover, it is shown how it can be decomposed into several interacting functional subsystems, each of them having the ability to express socio-economical functions and purposes.

Section 3 deals with the derivation of a specific mathematical model for the external action exerted by media. The derivation needs to be developed taking into account the specificity of the role of media action. The model is then inserted within the framework of Section 2.

Section 4 deals with the derivation of a specific model of competition for a secession. The derivation needs, as we shall see, a detailed modelling of the interactions between functional subsystems and among them and the outer environment, namely media. This section finally provides the derivation of a mathematical framework suitable to design specific models.

Section 5 develops simulations focused on analysing the predictive ability of the model with special attention to the role of the external action. Specifically, it is shown how these actions, which are supposed to operate on limited time intervals and have a fixed cost, can modify the output of the competition.

Section 6 critically analyzes some specific applications and provides an overview on research perspectives, and is also focused on further developments of the mathematical model in view of optimization problems.

2. Phenomenological description of the system: Competition for a secession under influence of media

This section deals with the phenomenological description of the socio-economical system under consideration, taking into account the fundamental role of external actions. We propose the same mathematical structure dealt with in [1], focusing on the discrete case.

Before approaching the mathematical modelling details, some introduction is needed. The recent socio-economical literature has shown an increasing interest in the development of models regarding complex political phenomena, as seen in [22–24], even if from a theoretical different approach, which avoid the concept of complex social dynamics. Some interesting references can be found in [25–27], where socio-political phenomena like dictatorships, terrorism and strategical interactions among interests groups are modelled and discussed from a quantitative approach, which takes into account dynamical interactions among agents.

Moreover, while the subject of media influence has been deeply treated in the modern socio-political literature, as witnessed by [28–31], the present literature is lacking the discussion of socio-economical phenomena under the influence of media, from a quantitative approach. The aim of this paper is to propose a first attempt to explain such a complex influence using the theory developed in [1].

The system, which we refer to, is a nation which, as visualized in Table 2.1, is decomposed into two or more subsystems identified by regional interest groups, which express, through specific actions taken by either their political parties or their peculiar interest groups, their attitude towards a process of secession by expressing a function u, which takes negative or positive values: when a certain subsystem is expressing a positive value of u, then it is pro-secession, when it is expressing a negative value of u, then it is against it. The absolute value of u measures the intensity of the function expressed. Another element which plays an important role in the system is the so-called external action, played by media: in our system, each interest group or party can be strongly affected by media, which can be viewed as an additional external functional subsystem; this means that media is another very powerful element of the decomposition.

Bearing this decomposition in mind, the overall nation is viewed as a network of interacting functional sub-systems, each of them corresponding to different interest groups and political parties. Their representation is based on the assumption that each of them has the ability to express a specific function, namely their attitude towards secession. Also the role of the media is initially modelled, considering it as a different kind of subsystem, which can exert an external action.

Let us consider a network of n interacting functional subsystems whose function is identified by the variable u, where the value $u = 0$ separates the positive and negative valued ***functions*** expressed by each subsystem. In particular, the socio-economical variable is identified by

$$I_u = \{u_{-p} = -1, \dots, u_0 = 0, \dots, u_p = 1\},$$

where the description of the state of each subsystem is delivered by the discrete probability density

$$f_i^j = f_i^j(t): \quad [0,T] \times \mathbb{R} \to \mathbb{R}_+^n, \quad \sum_{j=1}^{p} f_i^j(t) = 1, \quad \forall\, t \geq 0, \tag{1}$$

for $i = 1, \dots, n$ and $j = -p, \dots, p$.

We now focus on systems in which the number n of subsystems is constant and the number of possible socio-economical states is H: this means that we are not dealing with destructive or aggregative events. Interactions are supposed to be binary.

The mathematical framework to describe microscopic interactions between two subsystems can be described by means of two different functions:

- the ***encounter rate*** η_{ij}^{pq}, which depends on the interacting subsystems i and j and on the socio-economical states, p and q respectively, of the interacting particles.
- the ***transition probability density*** $\mathcal{B}_{ij}^{pq}(h) = \mathcal{B}_{ij}(u_p, u_q; u_h)$, that describes the probability density that a candidate particle with state u_p of the subsystem i falls into the state u_h after the interaction with a field particle of the subsystem j with state u_q.

The transition probability density function is such that:

$$\forall i, j, \quad \forall p, q: \quad \sum_{h=1}^{H} \mathcal{B}_{ij}^{pq}(h) = 1. \tag{2}$$

When, as we are doing, we consider open systems, that interact also with the outer environment, we have to deal with functional subsystems interacting with external agents identified with the subscript $\ell = 1, \dots, m$, where m is constant.

The external agent ℓ has the ability to influence the socio-economical state u of the subsystem i, through a particular action identified by the variable v. The analysis developed in what follows is based on the assumption that the action of the outer agents can be modelled as follows:

$$\nu_{i\ell}(t)\, g_\ell(v), \tag{3}$$

where $\nu_{i\ell} = \nu_{i\ell}(t)$ is the intensity, that can depend on time, during which the agent ℓ acts on the subsystem i; and $g_\ell(v)$ is the probability density associated to the variable v that characterizes the action of outer system, namely the probability that this action takes place over system i.

Notice that each external agent is regarded as a specific functional sub-system with the ability to interact with the functional sub-systems of the inner system. Both terms $\nu_{i\ell}$ and g_ℓ are supposed to be given functions of their arguments.

The mathematical framework to describe the microscopic interactions between a subsystem i and an external agent ℓ needs the following discrete interaction terms:

- The ***inner–outer encounter rate*** $\mu_{i\ell}$ that describes the rate of interactions between an external agent ℓ and a subsystem i and does not depend on the socio-economical state of the interacting particles.
- The ***inner–outer transition probability density*** $\mathcal{G}_{i\ell}^{pq}(h) = \mathcal{G}_{i\ell}^{pq}(v_q, u_p; u_h)$, that describes the probability density that a candidate particle with state u_p of the subsystem i falls into the state u_h after the interaction with a field particle of the external agent ℓ with state v_q.

The interaction term $\mathcal{G}_{i\ell}^{pq}(v_q, u_p; u_h)$ satisfies the following condition:

$$\forall i, \ell, \quad \forall p, q: \quad \sum_{h=1}^{H} \mathcal{G}_{i\ell}^{pq}(h) = 1. \tag{4}$$

Technical calculations yield:

$$\frac{\mathrm{d}f_i^h}{\mathrm{d}t} = \sum_{j=1}^{n} \left(\sum_{p=1}^{H} \sum_{q=1}^{H} \eta_{ij}^{pq} \mathcal{B}_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=1}^{H} \eta_{ij}^{pq} f_j^q \right) + \sum_{\ell=1}^{m} \left(\sum_{p=1}^{H} \sum_{q=1}^{H} \nu_{i\ell} \mu_{i\ell}^{pq} \mathcal{G}_{i\ell}^{pq}(h) g_\ell^q f_i^p - f_i^h \sum_{q=1}^{H} \nu_{i\ell} \mu_{i\ell}^{pq} g_\ell^q \right). \tag{5}$$

Eq. (5) can be regarded as the fundamental framework for the derivation of models as we shall see in the next section.

3. Modelling external actions

While the reasonings of Section 2 are very general and they can be applied to many phenomena, now we want to define the external action in a very specific case, namely the "competition for a secession" phenomenon under the influence of media.

In these last centuries, the role of communications becomes more and more important: it is well known that social and political events are strongly affected by media, this is the reason why a development of the model in [1] is necessary, introducing this important external role. Before approaching the mathematical modelling part, some introduction is required.

In real economic world, any action has a cost. This means that if media want to influence everyday phenomena, they have to design specific strategies under their budget constraints. In particular, if television, radio and newspaper want to modify the political life of a country, they have to do it, under specific budget limitations. Moreover, they have to choose which kind of influence they want to exert and which kind of purpose they want to reach. From now on, we will call this behavior a "strategy": once a specific purpose is designed, media have to choose a strategy to reach this purpose at their best. For example, if the owner of a big newspaper wants to influence the population of a country pro or against secession, he or she has to decide whether to invest his/her budget towards influencing political parties or to split the same budget among the different interest groups in the country. In addition, we can think of media action to be decreasing in time. This means that media influence lasts for just a certain window of time T_s, given a certain level of investment, and, after this period, expires, so that interactions are the same as without it. Clearly, the bigger the investment, the longer the influence.

Notice that the mathematical derivation of this section refines the one described in the previous section, since we derive the specific action of external agents, in the case of media.

The different external agents are denoted by the subscript ℓ. The external agent's influence is represented by

$$\nu_{i\ell}(t) = \varepsilon_{i\ell} a_i(t). \tag{6}$$

The action of media is considered certain, therefore we do not take into account any probabilistic influence of media towards subsystems: if subsystems encounter media action they will be affected anyway. The action of the external agent is affecting the different level of subsystem opinions, identified by u.

Every external agent ℓ can influence the subsystem i, with an intensity $\varepsilon_{i\ell}$: this is the action over the state of the opinion u_p. This means that by media actions, the opinion distribution of the subsystem i can change. $a_i(t)$, instead, represents the duration of such an influence and it is just an action over time.

Following the previous introduction, the level of intensity expressed by the external agents ℓ, is subjected to a budget constraint:

$$\sum_{i,\ell} \varepsilon_{i\ell} \int_0^{T_s} a_i(s)\mathrm{d}s = C_{\mathrm{tot}}, \tag{7}$$

where C_{tot} represents the maximum cost the media owner can sustain, for the time interval T_s. Notice that the cost C_{tot} corresponds to the level of investment the media owner must afford to influence opinions during a time period T_s. For $t > T_s : a_i(t) = 0$. The total cost can be normalized so that $C_{\mathrm{tot}} = 1$.

Notice that the action in time is exerted in the interval $[0, T_s]$, which can, from a mathematical point of view, be normalized to $[0, 1]$. Nevertheless, in the next section we are going to investigate the role of T_s in the general applied phenomenon, even when it exceeds the unitary value.

The purpose of the external agent, given a fixed budget, consists in finding the best repartition of intensities to influence opinions, in order to choose which opinions the efforts have to be concentrated on. In particular, $\varepsilon_{i\ell}$ is constant in time and varies with u, namely the intensity of the action is distributed over different opinions, according to the different strategy chosen. It is possible to use all the intensity to influence opinion u_h or, instead, to use the same intensity divided by n over all the opinions $u_1, \ldots, u_n$.

$a_i(t)$ represents the duration of the influence and it can assume different forms: it can be a function constant over time, or a function decreasing in time, depending on which kind of influence we want to model.

In the next section we will focus on the case in which a_i is such that $\int_0^{T_s} a_i(s)\mathrm{d}s = T_s$, referring to a case of interest for the applications. In this case we just have to focus on the values of ε_{ih}.

4. On the competition model and strategy

Let us now consider the modelling of a competition for a secession, referred to a system corresponding, in this specific problem, to a nation subjected to the media action: i. e. a nation where broadcasting television companies are free to operate. We assume u to be a discrete variable, since it better represents the different level of opinions, related to each group (subsystem). The domain interval $[-1, 1]$ is defined as follows

$$I_u = \{u_1 = -1,\ u_2 = -0.5,\ u_3 = 0,\ u_4 = 0.5,\ u_5 = 1\} \tag{8}$$

representing H different level of opinions.

Let us assume that the nation is divided into a richer and a poorer region, so that the richer part hopes to benefit from a secession in terms of national income and taxes. For simplicity, we suppose that there is just a single television network, owned by someone living in the richer region, wanting to manipulate the global information in order to favor secession. Many cases of everyday reality in which information manipulation seems to deeply influence the political scenario of a country can be figured. Differently oriented political parties and interest groups interact in the global nation in order to support the secession or not and they have to deal with media information. The specific model proposed in what follows is characterized by a small number of variables and very simple interaction functions. The aim is to show that, even in a simple case, the model can describe some interesting phenomena. Following the approach proposed in [1], the first step of the modelling consists in identifying the functional subsystems that compose the overall system. Accordingly, let us suppose to decompose the system into the following subsystems:

- Political parties,
- Unions,
- White collars,
- Broadcasting TV company exerting an external action.

Let us focus on the above four subsystems to define a simple model suitable to describe the evolution of the system using a small number of parameters. Bearing in mind that each subsystem can be anyway represented in a much more complex way, considering the different orientation (left, right and center) that are present in political parties, unions and white collars, and the varieties of influence which a television system can exert, we define the following discrete probability densities, regarding the three internal subsystem, since we suppose the action of media to be certain:

- $f_1^h(t)$, describes the distribution over the state of the subsystem ***political parties***,
- $f_2^h(t)$, describes the distribution over the state of the subsystem ***unions***,
- $f_3^h(t)$, describes the distribution over the state of the subsystem ***white collars***.

The second step consists in modelling the interaction terms, namely the matrix of the encounter rates η_{ij} among unions, white collars and political parties, the matrix of the inner–outer encounter rates $\mu_{i\ell}$, among subsystems and media, the parameters of the transition probability density and the intensities regarding the external action of TV $\varepsilon_{i\ell}$. Notice that $\ell = 1$, since we just deal with a single external agent.

Let us deal with the modelling of the encounter rate η_{ij} according to the following assumption:

Hp. 1. *$\eta_{ij} = \eta_{ji}$ does not depend on the state of the interacting subsystems, but only on the interacting pairs.*

In particular we suppose that η_{ij} attains the largest value when the interactions takes places within the same subsystems: $\eta_{11} = \eta_{22} = \eta_{33} = \eta_0 = 1$, while $\eta_{12} = \eta_{21} = \alpha_1 < \eta_0$, $\eta_{13} = \eta_{31} = \alpha_2 < \alpha_1\eta_0$, $\eta_{23} = \eta_{32} = \alpha_3\eta_0$, where $\alpha_2 < \alpha_3 < \alpha_1$.

The values of η_{ij} are reported in the following matrix:

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ \alpha_1 & 1 & \alpha_3 \\ \alpha_2 & \alpha_3 & 1 \end{pmatrix}. \tag{9}$$

Referring to the inner–outer encounter rate $\mu_{i\ell}$, we simply assume that the citizens of the country watch TV uniformly, independently from their social status: $\mu_{i\ell} = \gamma$, for $i = 1, 2, 3$.

Let us now model the discrete transition probability density function. The modelling is developed according to the following assumption consistent with the approach proposed in the previous section.

The discrete probability density function, $\mathcal{B}_{ij}^{pq}(h) = \mathcal{B}_{ij}(u_p, u_q; u_h)$, represents the probability density that a particle with state u_p of the population i falls in the state u_h, after the encounter with a particle with state u_q of the population j. We look for a discrete function such that

$$\forall i, j, \quad \forall p, q \; : \quad \sum_{h=1}^{5} \mathcal{B}_{ij}^{pq}(h) = 1. \tag{10}$$

Let us introduce a *critical distance d_c*, such that, if encounters take place between individuals, whose opinions are closer than d_c, then the two parties get nearer according to their opinions (*dialectic case*); if encounters take place between individuals, whose opinions are more distant than d_c, then the two parties get farther according to their opinions (*non dialectic case*). This means that the model can become either dialectic or non dialectic depending on the distance in state of the interacting subsystems: if d_c is big enough the parties can reach an agreement while if d_c is not, there is a conflict solution. Notice that it is possible to define more than a single critical distance d_c, so that there are many intervals in the real line in which the model is dialectic and others in which it is not.

We can define the discrete transition probability density function as follows. If $p = q \Rightarrow \mathcal{B}_{ij}^{pq}(h = p) = 1$, $\mathcal{B}_{ij}^{pq}(h \neq p) = 0$. If $p \neq q$, $| p - q | \leq d_c$, the model is dialectic, namely, if $p < q$ one has

$$\mathcal{B}_{ij}^{pq}(h = p + 1) = \beta, \qquad \mathcal{B}_{ij}^{pq}(h = p) = 1 - \beta, \qquad \mathcal{B}_{ij}^{pq}(h \neq p, p + 1) = 0;$$

if $p > q$:

$$\mathcal{B}_{ij}^{pq}(h = p - 1) = \beta, \qquad \mathcal{B}_{ij}^{pq}(h = p) = 1 - \beta, \qquad \mathcal{B}_{ij}^{pq}(h \neq p, p - 1) = 0.$$

If $p \neq q$, $|\, p - q \,| \geq d_c$, the model is non dialectic, namely, if $p < q$

$$\mathcal{B}_{ij}^{pq}(h = p - 1) = \beta, \qquad \mathcal{B}_{ij}^{pq}(h = p) = 1 - \beta, \qquad \mathcal{B}_{ij}^{pq}(h \neq p, p - 1) = 0;$$

if $p > q$

$$\mathcal{B}_{ij}^{pq}(h = p + 1) = \beta, \qquad \mathcal{B}_{ij}^{pq}(h = p) = 1 - \beta, \qquad \mathcal{B}_{ij}^{pq}(h \neq p, p + 1) = 0,$$

where $\beta > 1/2$.

Let us now model the external action over the inner subsystem i. According to (6), the influence of media ℓ over subsystem i can be thought as $\nu_{i\ell} = \varepsilon_{i\ell} a_i(t)$, where $\varepsilon_{i\ell}(u)$ represents the intensity of the action and $a_i(t)$ its the duration. For simplicity, we assume that for all i one has:

$$a_i(t) = \begin{cases} 1 & \text{if } t < T_s, \\ 0 & \text{if } t \geq T_s. \end{cases} \tag{11}$$

This means that the action of TV is constant in time for a duration T_s, depending on the level of investment. The mathematical modelling of the intensities $\varepsilon_{i\ell}$ is more complicated and it is related to the concept of strategy. As we stated in the previous paragraph, we assume

$$T_s \sum_{i=1}^{3} \sum_{p=1}^{5} \varepsilon_{i\ell}(p \to h) = C_{\text{tot}},$$

where $\varepsilon_{i\ell}(p \to h)$ represents the external action which transforms opinion p into the new opinion h.

We consider C_{tot} fixed. Moreover, we consider the case in which the media action concentrates in order to influence citizens pro secession. There are many explanations about it such as,a media owner is in favor of separation since he lives in the richer region or he thinks he could benefit from the separation, reducing the TV market and so on.... According to this scenario the TV owner can chose different strategies to reach his purpose: we will investigate three of them.

Let us imagine that opinion 1 is the radical opinion against secession, opinion 5 the radical one towards secession and the other opinions the moderate ones, pro or against it. We can model the budget constraint assuming that TV shifts opinions for a maximum of 4 hops for every subsystems: this means that TV owner can decide whether to influence just individuals with opinion 1 towards opinion 5 or to influence individuals of every opinion, just shifting them to one single opinion. In both cases the total number of hops that TV generates is 4, since we assume that people with opinion 5 cannot move further.

Bearing this in mind, we define the three different strategies to investigate in the next section. Notice that the aim of the investigation is to explore different possible strategies in order to understand and compare them, referring to the final outcome. Simulation are to be conceived in this sense.

- *Strategy 1*: TV concentrates all its efforts upon opinion 1: when individuals with opinion 1 encounter media, they radically change their mind and become individuals of opinion 5,
- *Strategy 2*: TV concentrates its efforts upon central opinions 2 and 3: when individuals with opinions 2 and 3 encounter TV, they respectively go towards opinion 4 and 5,
- *Strategy 3*: TV concentrates its efforts upon every opinion: when individuals with opinion p encounter TV, they respectively go towards opinion $p + 1$, except for individuals with opinion 5 who do not change opinion.

More in details we define the external action for the three strategies as follows for $i = 1, 2, 3$:

- **Strategy 1**: $\varepsilon_{i\ell}(u_p) = u_5$ if $p = 1$ and $\varepsilon_{i\ell}(u_p) = 0$ if $p \neq 1$;
- **Strategy 2**: $\varepsilon_{i\ell}(u_2) = u_4$ and $\varepsilon_{i\ell}(u_3) = u_5$, $\varepsilon_{i\ell}(u_p) = 0$ if $p \neq 2, 3$;
- **Strategy 3**: $\varepsilon_{i\ell}(u_p) = u_{p+1}$, $\forall p = 1, 2, 3, 4$, $\varepsilon_{i\ell}(u_p) = 0$ if $p = 5$.

Taking into account the previous definitions, it is possible to obtain the following 3×5 systems of ordinary differential equations:

$$\frac{df_i^h}{dt} = \sum_{j=1}^{3} \left(\sum_{p=1}^{5} \sum_{q=1}^{5} \eta_{ij} \mathcal{B}_{ij}^{pq}(h) f_i^p f_j^q - f_i^h \sum_{q=1}^{5} \eta_{ij} f_j^q \right) + a_i(t) \left(\sum_{p=1}^{5} \varepsilon_{i\ell}(p \to h) \mu_{i\ell} f_i^p - f_i^h \varepsilon_{i\ell}(h \to p) \mu_{i\ell} \right), \tag{12}$$

where $i = 1, 2, 3$ and $h = 1, 2, 3, 4, 5$, $\varepsilon_{i\ell}(p \to h)$ represents the external action over f_i^p which transforms opinion p into opinion h and $\varepsilon_{i\ell}(h \to p)$ the external action over f_i^h which transforms opinion h into a different opinion.

5. Simulations

This section develops an analysis finalized to the selection of the optimal strategy, based on simulations which refer to each strategy.

These simulation are obtained by solving the initial value problems generated by model (12) linked to suitable initial conditions.

In particular, referring to the model described in the previous section, simulations are developed to obtain the time evolution of the three subsystems under consideration, respectively political parties, unions and white collars. The following parameters representing the encounter rate and the transition probability density: $\alpha_1 = 0.75$, $\alpha_3 = 0.45$, $\alpha_2 = 0.25$, $\mu_1 = \mu_2 = \mu_3 = 1$ and $\beta = 1$, are adopted, while the interval of opinions, spanning in the interval [0, 1], is discretised into 5 different levels, representing the 5 different opinions which can be assumed within each subsystem.

Keeping all these parameters fixed, the goal of the simulations is to investigate the role of the two fundamental parameters of the system: the critical distance d_c and the media time action T_s, in the evolution of f_1, f_2 and f_3, given random uniformly distributed initial conditions. Simulations are developed in the case in which the two extreme opinions are free to move between opinion 0 and opinion 1. All the figures reported hereby are such that

- *opinion 1* corresponds to continuous line
- *opinion 2* corresponds to dashed line
- *opinion 3* corresponds to thick line
- *opinion 4* corresponds to dot-dashed line
- *opinion 5* corresponds to very thick line.

Each strategy is analysed singularly. Subsequently a comparison among them is developed.

5.1. Strategy 1

Media implement strategy 1 when all actions are concentrated upon opinion 1, which is directly shifted to opinion 5. Even in this very preliminary case, results are very different from the case of no media action.

When $d_c = 0, 1$ and $T_s < 3$, the action of media has no influence in the evolution and results are similar to the ones in [1]: we can see that for $t < T_s$ opinion 1 is decreasing, then it is possible to notice an angle point for $t = T_s$, and, for $t > T_s$, opinion 1 increases its value towards the equilibrium, as reported in Figs. 1–3. This means that for low value of T_s media cannot influence people's opinion: when media action takes place, namely for $t < T_s$, opinion 1 is decreasing its value because of media strategy. For $t > T_s$, instead, media action is expired and opinion 1 inverts its trend, starting to increase its values. For $T_s = 3, 4$ a transition emerges: opinion 1 has a much smaller intensity at equilibrium than for $T_s < 3$; for $T_s \geq 5$ the transition takes place, namely opinion 1 survives with very little intensity, while opinion 5 has significant values, as shown in Figs. 4–6. Notice that, starting from this very preliminary case, results are much different from those reported in [1], where for $d_c = 0, 1$ opinion 1 and 5 were surviving with almost the same intensity.

When $d_c = 2$, we observe a slow convergence and the so-called compromise solution emerges. When $T_s < 10$, opinion 5 can survive together with opinion 4 or opinion 4 can survive together with opinion 3, depending on initial conditions. For values of T_s near 10 convergence is still slow and, in particular, much slower than in the no media case. The trend is however evident: opinion 4 and 5 survive sometimes together sometimes alone, depending on initial conditions. More rarely there is the survivance of opinion 4 with opinion 3. Notice that, again, this is much different from the case with no media: if $d_c = 2$ people are less extreme than when $d_c = 0, 1$ but are more right oriented than in [1]; thanks to media action the population is anyway shifted towards opinion 5 or close opinions, namely opinion 4, which can be considered as a "moderate extreme opinion". The moderate equilibrium behavior is clearly due to d_c. See Figs. 7–12.

For $d_c = 3, 4$, even for small values of T_s, opinion 2 never survives, just opinion 3 and 4 survive, depending on initial conditions. Again the trend is to have a population shifted towards 5, as shown in Figs. 13–15.

To summarize, T_s is a key parameter for low values of d_c: d_c still plays a fundamental role since it measures the tendency of people to be extreme; anyway the media influence is remarkable, since, for suitable values of T_s, all the population is shifted towards the opinion desired by media. In particular, we can see from figures that, when media action takes place, opinion 1 is always decreasing, for $t > T_s$ it can raise a little, but the trend is however established. If people tend to be extremists (low values of d_c) just opinion 5 survive, while if people tend to agree (high values of d_c) the surviving opinion is mainly opinion 4: strong evidence of media influence.

5.2. Strategy 2

Strategy 2 concentrates upon two different opinions, opinion 2 and opinion 3, shifting them respectively into opinion 4 and 5. In this case, for $d_c = 0, 1$ the parameter T_s has no influence in the evolution of the three densities: opinions 1 and 5 survive as in the case with no media action, the unique difference is that opinion 1 is rarely the dominant one, see Figs. 16–18.

For $d_c = 2$, solutions are very similar to the one for $d_c = 0, 1$, except for the survivance of opinion 4 for a certain window of time and for the increasing value of opinion 5, which is higher than before. Again T_s has no influence, see Figs. 19–21.

When $d_c = 3$, the transition has taken place: opinion 3 and 4 survive even for $T_s = 1$ and opinion 4 is generally more influent that opinion 3. Notice that for $t = T_s$ there is an angle point which shows that for $t < T_s$ opinion 3 is decreasing due to media action and for $t > T_s$ it starts increasing, since media action is expired. When $T_s = 2$ opinion 4 begins to appear with opinion 5. This phenomenon is more evident for $T_s > 3$. Notice that this is much different from the no media case. When $d_c = 4$ the phenomenon is the same as before, just more evident: for $T_s = 1$ opinion 3 and 4 survive, then for $T_s = 2$ just opinion 4 survives and for $T_s \geq 3$ opinion 4 survives together with opinion 5. See Figs. 22–24.

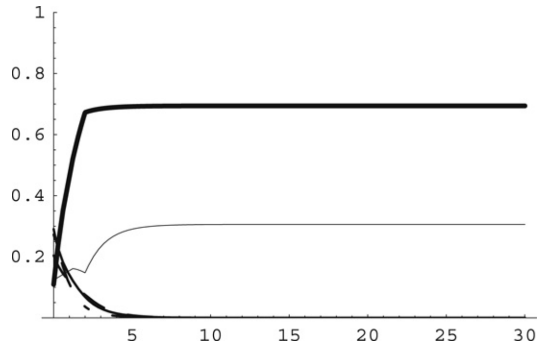

Fig. 1. Strategy 1 — evolution of the components of f_1 in the case of $d_c = 0, 1$, $T_s = 2$.

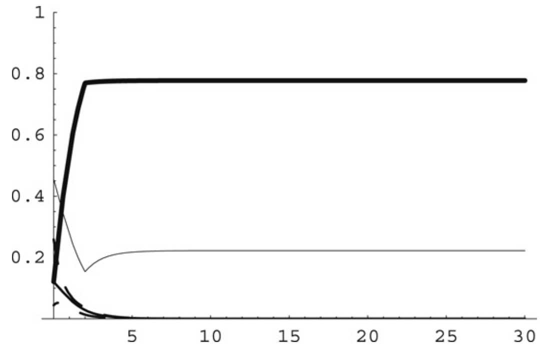

Fig. 2. Strategy 1 — evolution of the components of f_2 in the case of $d_c = 0, 1$, $T_s = 2$.

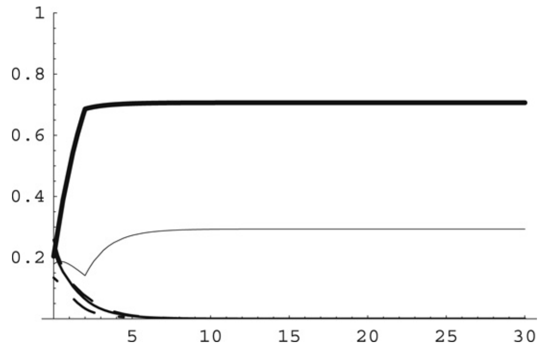

Fig. 3. Strategy 1 — evolution of the components of f_3 in the case of $d_c = 0, 1$, $T_s = 2$.

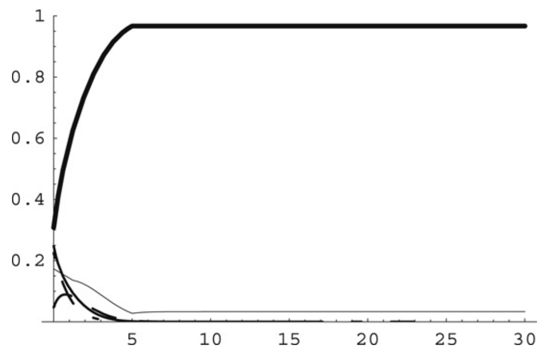

Fig. 4. Strategy 1 — evolution of the components of f_1 in the case of $d_c = 0, 1$, $T_s = 5$.

Even in this case the differences between the case with no media are absolutely evident. Opinion 3 almost never survives and all the population is shifted towards opinion 5 depending on the critical distance d_c. In particular, it seems that for low

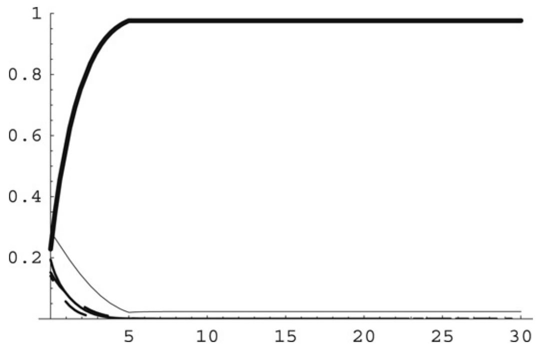

Fig. 5. Strategy 1 — evolution of the components of f_2 in the case of $d_c = 0, 1, T_s = 5$.

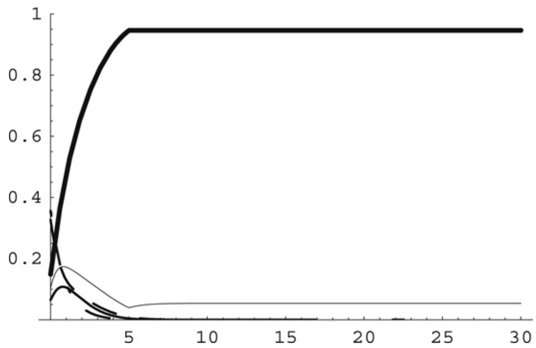

Fig. 6. Strategy 1 — evolution of the components of f_3 in the case of $d_c = 0, 1, T_s = 5$.

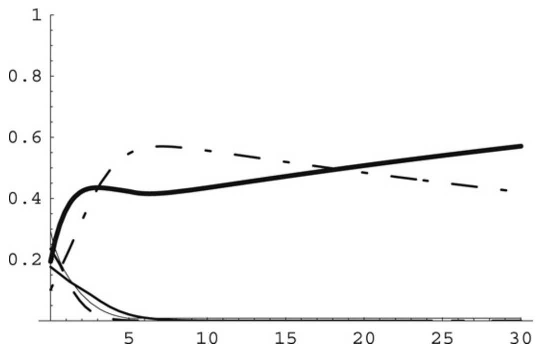

Fig. 7. Strategy 1 — evolution of the components of f_1 in the case of $d_c = 2, T_s = 5$.

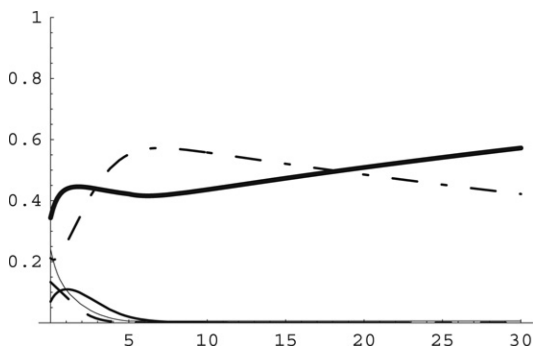

Fig. 8. Strategy 1 — evolution of the components of f_2 in the case of $d_c = 2, T_s = 5$.

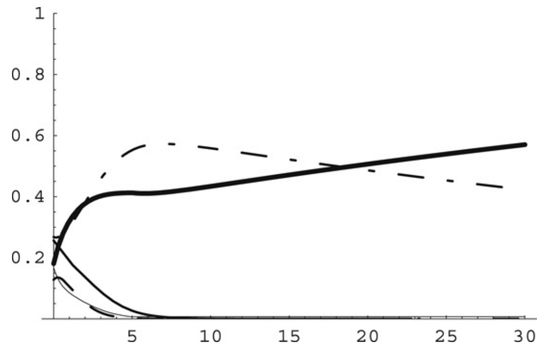

Fig. 9. Strategy 1 — evolution of the components of f_3 in the case of $d_c = 2$, $T_s = 5$.

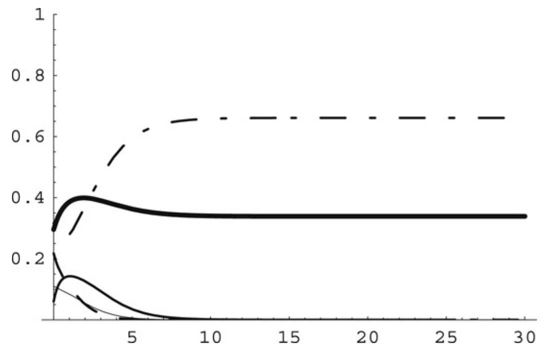

Fig. 10. Strategy 1 — evolution of the components of f_1 in the case of $d_c = 2$, $T_s = 15$.

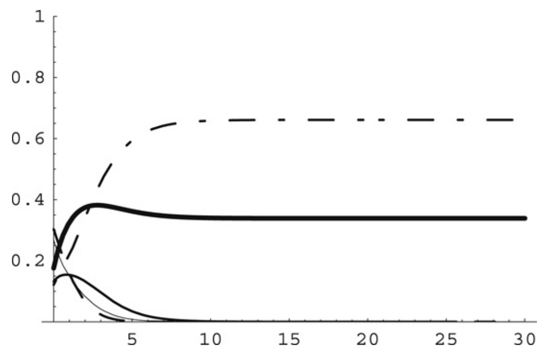

Fig. 11. Strategy 1 — evolution of the components of f_2 in the case of $d_c = 2$, $T_s = 15$.

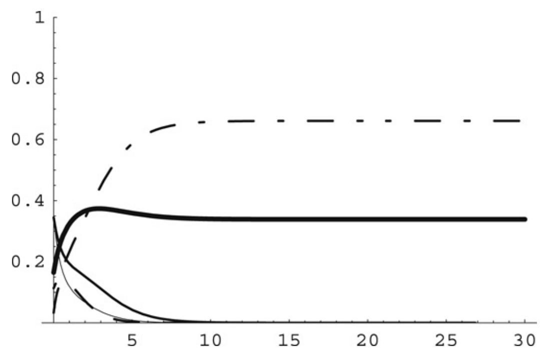

Fig. 12. Strategy 1 — evolution of the components of f_3 in the case of $d_c = 2$, $T_s = 15$.

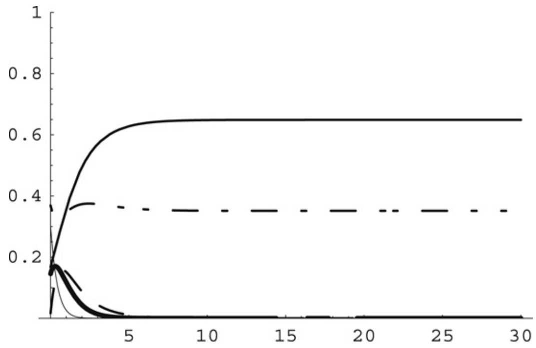

Fig. 13. Strategy 1 — evolution of the components of f_1 in the case of $d_c = 3, 4$, $T_s = 5$.

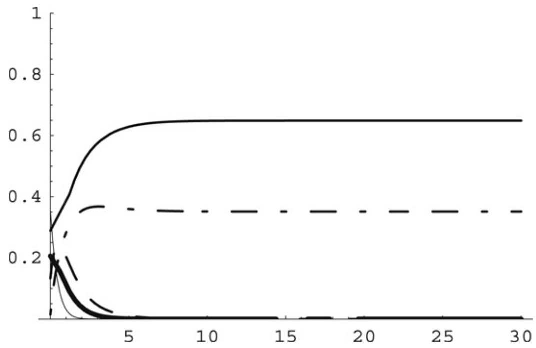

Fig. 14. Strategy 1 — evolution of the components of f_2 in the case of $d_c = 3, 4$, $T_s = 5$.

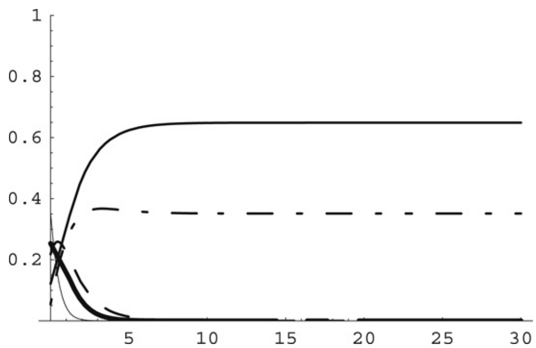

Fig. 15. Strategy 1 — evolution of the components of f_3 in the case of $d_c = 3, 4$, $T_s = 5$.

values of d_c, namely $d_c = 0, 1, 2$ this strategy is not very efficient since opinion 1 is still surviving. While for $d_c = 3, 4$ the strategy is really efficient, since, for a sufficient time exposition to media action, almost all the population is captured by opinion 5 or 4. This is reasonable since, in this case, the strategy acts on intermediate opinions, that are the ones which play an important role when d_c is high, if instead d_c is low, intermediate opinion tend to be attracted by extreme ones and the strategy has no power.

5.3. Strategy 3

Strategy 3 is such that opinion 1, 2, 3, 4 go respectively to opinion 2, 3, 4, 5 and opinion 5 remains the same. This strategy is the most uniform one in the sense that all the opinions are affected by media action.

When $d_c = 0, 1$, opinions 1 and 5 survive even if opinion 5 is much more intense. In this case media action response is highly evident and progressive, depending on T_s: the more T_s, the more opinion 5 has influence and opinion 1 decreases, as shown in Figs. 25–30.

When $d_c = 2$ the scenario is different: for $T_s \leq 4$, opinion 5 is very influent, even if surviving with opinion 1 and 4 for small values of t. For $T_s \geq 4$, opinion 5 is the unique to survive with a certain intensity. See Figs. 31–36.

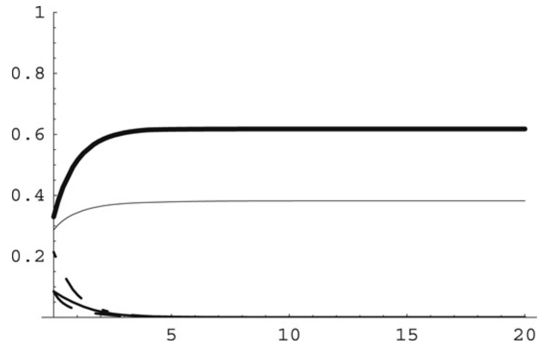

Fig. 16. Strategy 2 — evolution of the components of f_1 in the case of $d_c = 0, 1, T_s = 4$.

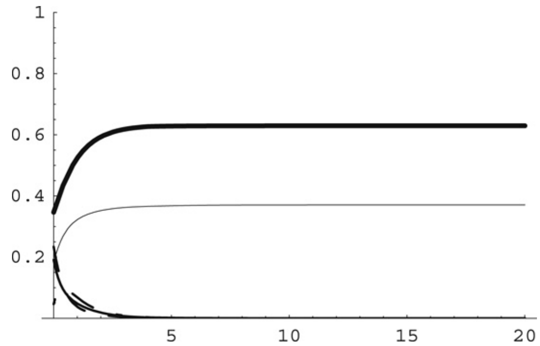

Fig. 17. Strategy 2 — evolution of the components of f_2 in the case of $d_c = 0, 1, T_s = 4$.

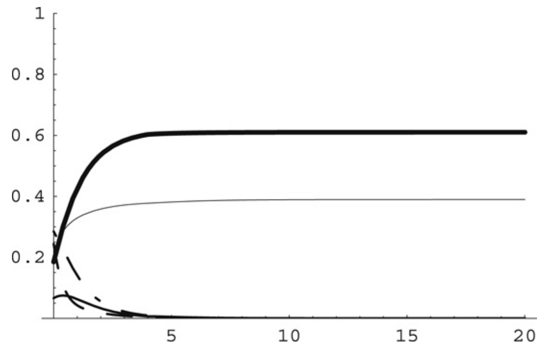

Fig. 18. Strategy 2 — evolution of the components of f_3 in the case of $d_c = 0, 1, T_s = 4$.

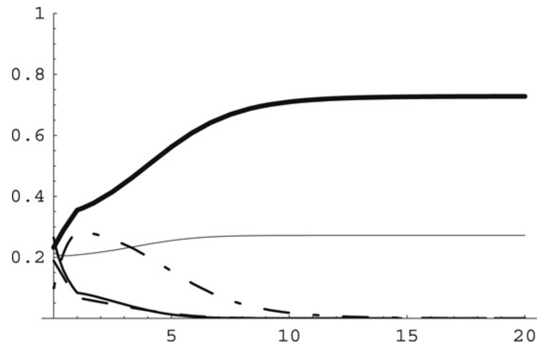

Fig. 19. Strategy 2 — evolution of the components of f_1 in the case of $d_c = 2, T_s = 1$.

When $d_c = 3$, the surviving opinions are 3 and 4 or 4 and 5 depending on initial condition, for $T_s = 1$. When $T_s = 2$, opinion 3 is no longer surviving and, for $T_s \geq 6$, opinion 5 is the unique to survive. If $d_c = 4$ results are almost the same

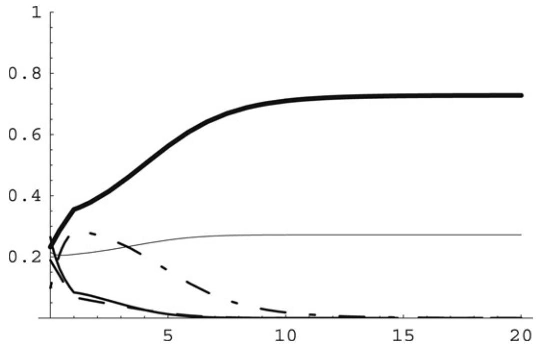

Fig. 20. Strategy 2 — evolution of the components of f_2 in the case of $d_c = 2$, $T_s = 1$.

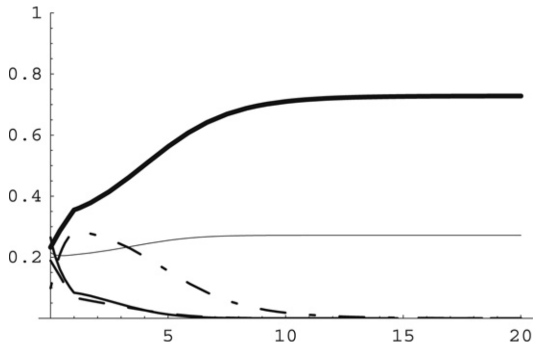

Fig. 21. Strategy 2 — evolution of the components of f_3 in the case of $d_c = 2$, $T_s = 1$.

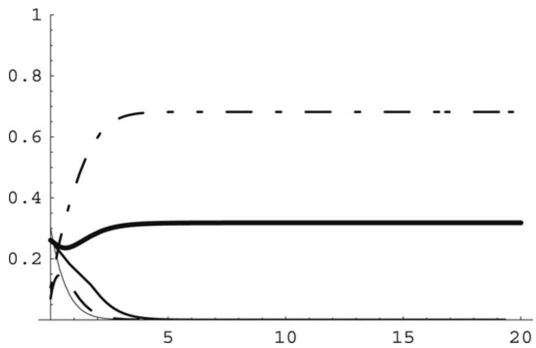

Fig. 22. Strategy 2 — evolution of the components of f_1 in the case of $d_c = 3, 4$, $T_s = 10$.

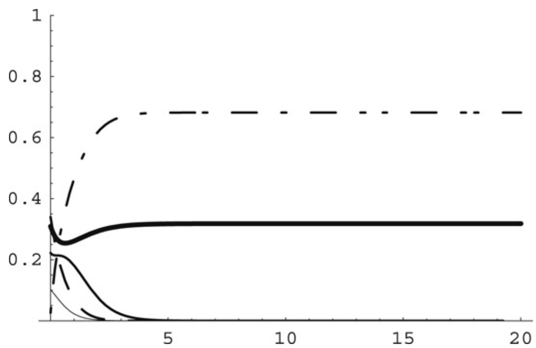

Fig. 23. Strategy 2 — evolution of the components of f_2 in the case of $d_c = 3, 4$, $T_s = 10$.

but just for $T_s \geq 4$ opinion 5 is the one which survives, with almost intensity 1, this means that almost all the population is influenced. See Figs. 37–42.

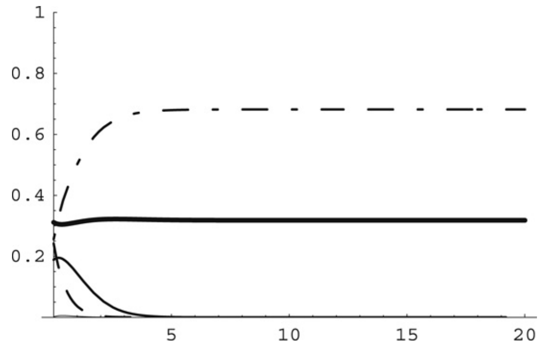

Fig. 24. Strategy 2 — evolution of the components of f_3 in the case of $d_c = 3, 4$, $T_s = 10$.

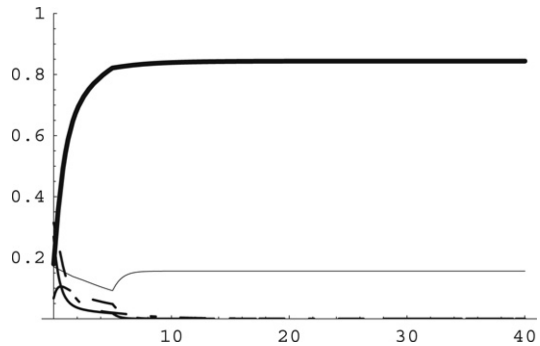

Fig. 25. Strategy 3 — evolution of the components of f_1 in the case of $d_c = 0, 1$, $T_s = 5$.

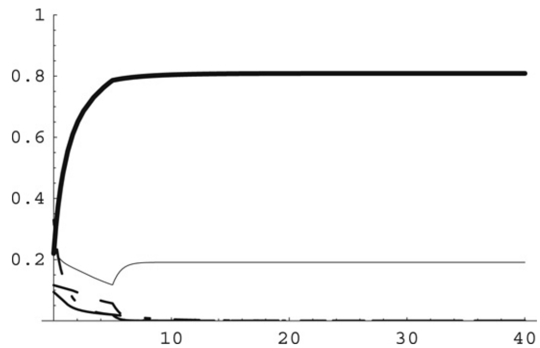

Fig. 26. Strategy 3 — evolution of the components of f_2 in the case of $d_c = 0, 1$, $T_s = 5$.

This strategy is very efficient, especially for $d_c = 3, 4$, even with a low value of T_s; however, it is possible to see a very good dependence from media action even for $d_c = 0, 1, 2$. With a sufficient value of T_s the goal of media is definitely reached, since almost all the population is captured by the desired media opinion.

6. Analysis of strategies and perspectives

A comparison among the various simulations is finally developed with the aim of focusing on the emerging events, due to the action of media. First of all it is remarkably evident that all the three strategies significantly influence opinions dynamics: results in all the three cases are very different from the ones found in [1]. Moreover, the interesting issue is that all the three strategies are different from the other in their results, even reaching the common aim.

As far as strategy 1 is concerned, we notice that T_s plays an important role for low values of d_c, while, regarding strategy 2, T_s plays an important role for high values of d_c: this is explained by the fact that strategy 1 is very efficient in those cases in which extreme opinions are fundamental in the dynamics, while strategy 2 is efficient in those cases in which the intermediate opinions are important to determine the evolution. This means that these two strategy can be more or less efficient depending on the degree of radicalization of the initial population: if the population is not inclined to compromise, strategy 1 is better, while if it is, strategy 2 works well.

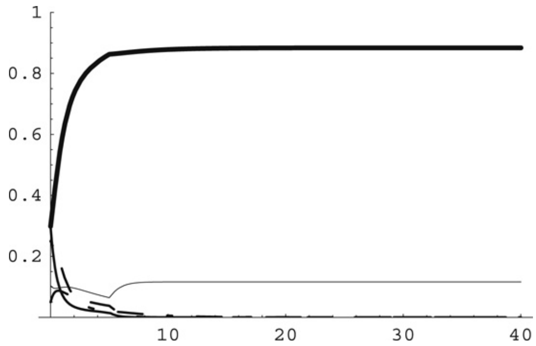

Fig. 27. Strategy 3 — evolution of the components of f_3 in the case of $d_c = 0, 1$, $T_s = 5$.

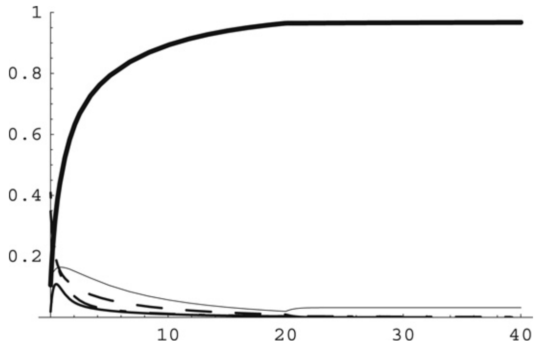

Fig. 28. Strategy 3 — evolution of the components of f_1 in the case of $d_c = 0, 1$, $T_s = 20$.

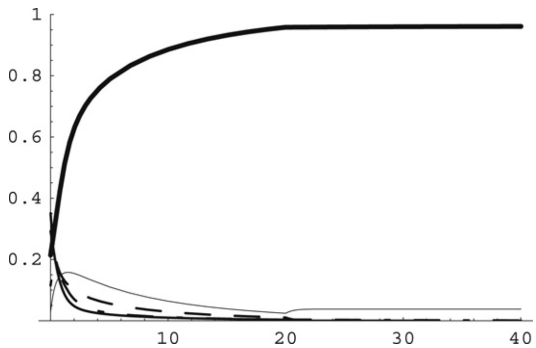

Fig. 29. Strategy 3 — evolution of the components of f_2 in the case of $d_c = 0, 1$, $T_s = 20$.

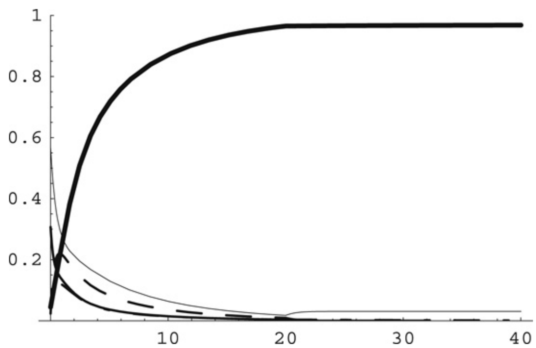

Fig. 30. Strategy 3 — evolution of the components of f_3 in the case of $d_c = 0, 1$, $T_s = 20$.

Anyway the most efficient results are obtained with strategy 3. In this case the dependence on T_s is strongly evident and, no matter which is d_c, there is always a time T_s, which guarantees to influence all the population towards opinion 5. This is

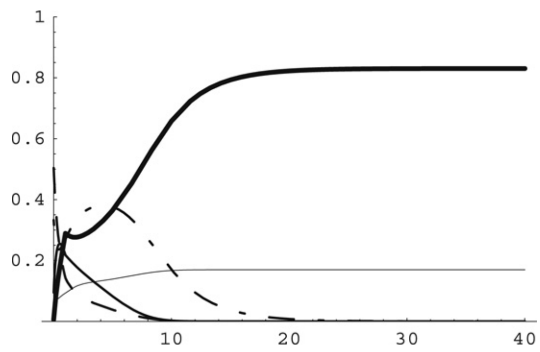

Fig. 31. Strategy 3 — evolution of the components of f_1 in the case of $d_c = 2$, $T_s = 1$.

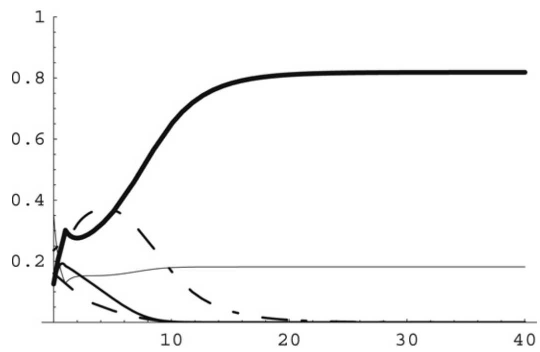

Fig. 32. Strategy 3 — evolution of the components of f_2 in the case of $d_c = 2$, $T_s = 1$.

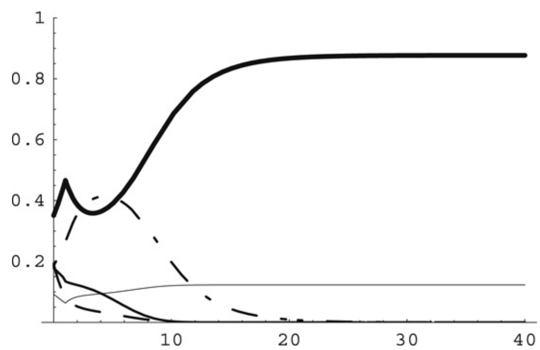

Fig. 33. Strategy 3 — evolution of the components of f_3 in the case of $d_c = 2$, $T_s = 1$.

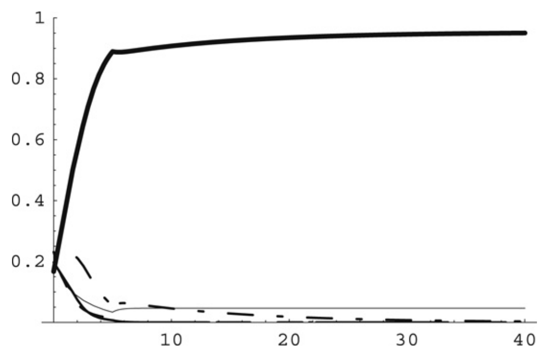

Fig. 34. Strategy 3 — evolution of the components of f_1 in the case of $d_c = 2$, $T_s = 5$.

even more evident for high values of d_c, when the dynamic is more free, since, due to people's interaction, all the population can be easily oriented to the direction settled by media. This is definitely the most efficient strategy.

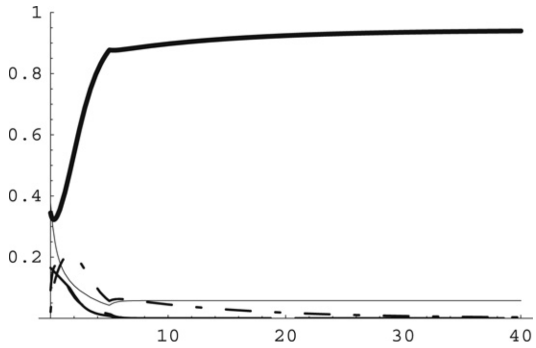

Fig. 35. Strategy 3 — evolution of the components of f_2 in the case of $d_c = 2$, $T_s = 5$.

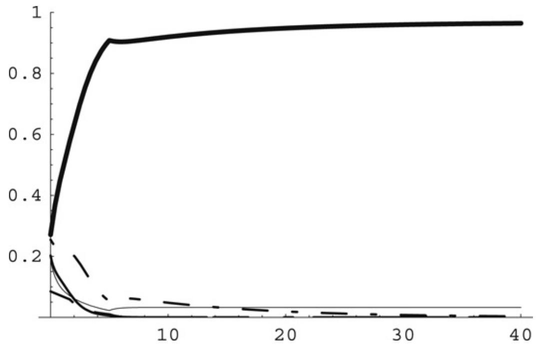

Fig. 36. Strategy 3 — evolution of the components of f_3 in the case of $d_c = 2$, $T_s = 5$.

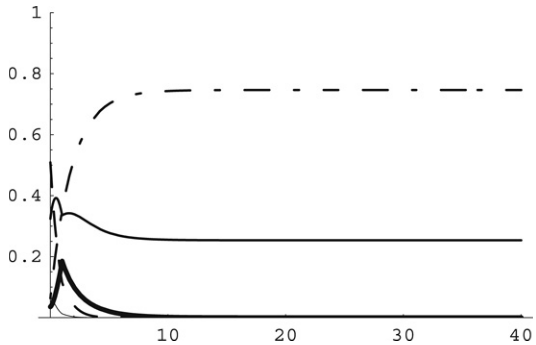

Fig. 37. Strategy 3 — evolution of the components of f_1 in the case of $d_c = 3, 4$, $T_s = 1$.

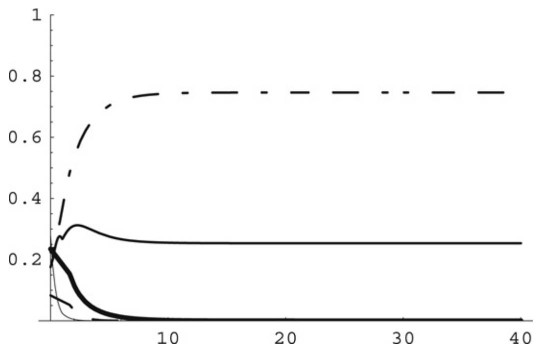

Fig. 38. Strategy 3 — evolution of the components of f_2 in the case of $d_c = 3, 4$, $T_s = 1$.

Summarizing, it seems, from simulations, that concentrating upon all opinions always leads to a winning outcome, while the choice to influence not all the opinions can lead to a favorable equilibrium, depending on the initial predisposition of the population.

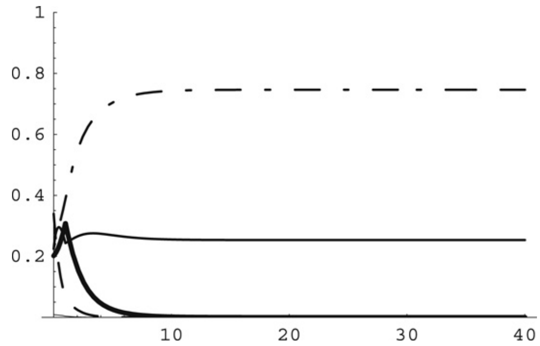

Fig. 39. Strategy 3 — evolution of the components of f_3 in the case of $d_c = 3, 4, T_s = 1$.

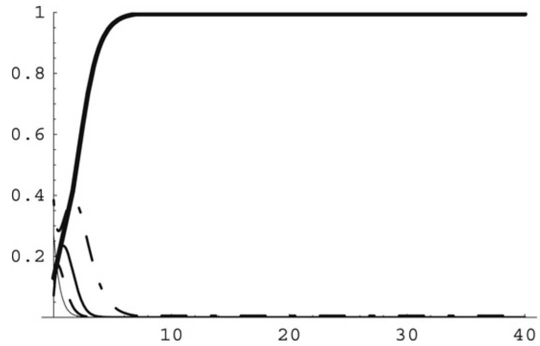

Fig. 40. Strategy 3 — evolution of the components of f_1 in the case of $d_c = 3, 4, T_s = 7$.

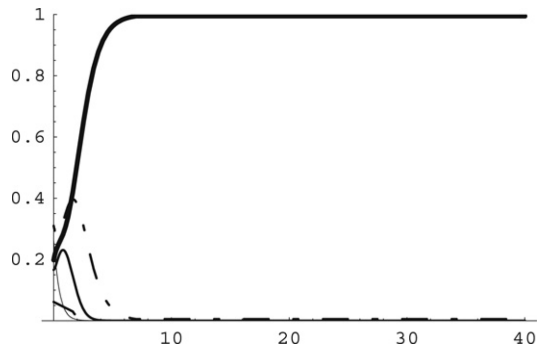

Fig. 41. Strategy 3 — evolution of the components of f_2 in the case of $d_c = 3, 4, T_s = 7$.

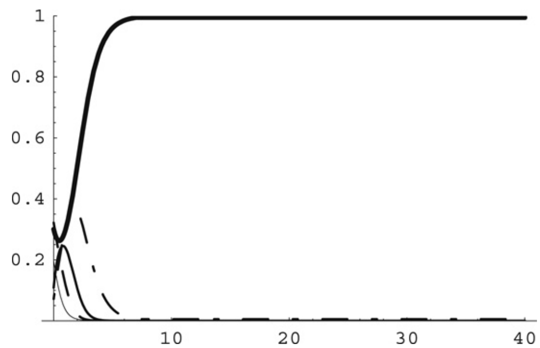

Fig. 42. Strategy 3 — evolution of the components of f_3 in the case of $d_c = 3, 4, T_s = 7$.

This paper has developed a detailed modelling and analysis of the competition for a secession phenomenon under the influence of media. The modelling is based on the approach proposed in [1], where two ingredients play an essential role in the modelling process. Namely the decomposition of the overall system into functional subsystems and the use of the approach of the kinetic theory for active particles. The modellings new contribution, with respect to [1] is the detailed description of external actions and, subsequently, the identification of specific strategies, which are supposed to satisfy the external action purposes. The identification of the optimal strategy is simply obtained by selecting, at fixed cost, the strategy that assures the most desired asymptotic behavior.

The contribution of this paper should be considered essentially methodological assuming that the approach can be technically generalized to different systems, such as those introduced in [1]. On the other hand, although the optimization problem can be put in a suitable analysis framework, the heuristic solution obtained by simulations appears to be useful and effective, since, even for a very preliminary model, they show peculiar emerging outcomes, which can be seen as the first attempt of further applied research developments.

Therefore, rather than focusing on research perspectives on analytic mathematical problems, it is worth stressing the utility of refining modelling aspects. Specifically, the assessment of the appropriate decomposition into functional subsystems and the identification of the strategies applied by the media.

Moreover, it is worth remembering that the model proposed hereby is a very preliminary attempt to formalize one of the most significant and important phenomena of our century. To investigate the topic further, it is absolutely necessary to develop a model where the media are more than one, expressing contrasting purposes and strategies at the same time. In this case it would be interesting to see which strategy is predominant and which is dependent on initial conditions.

Indeed, improving the above two issues definitely leads to improve the result of the optimization, keeping in mind that the methodological approach still remains the essential background for further developments and analysis.

To conclude, it is worth to remark that, the goal and the innovative aspect of this model is not only to give a detailed quantitative description of the phenomenon, while showing the emergence of a clear collective behavior, as a consequence of the interaction among functional subsystems and media.

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Fashion, an agent-based model

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Abstract—This paper deals with the development of an agent based model, regarding the phenomenon of fashion. It wants to investigate how fashion propagates among agents, when different types of dynamics are defined. Two classes of agents are introduced: common agents and fashion gurus (trend setters), having different roles and playing different strategic behaviors. A sensitivity analysis of some parameters of the model, by means of numerical simulations of the dynamic behavior of agents, concludes.

Keywords— Social dynamics, agent based models, fashion, trend-setters.

1 Introduction

Fashion is an intriguing phenomenon: it has been studied for a long time, in different fields of social sciences, with different aims and perspectives. Since it is a highly interdisciplinary subject of investigation it has risen the attention of scholars of many different fields, like sociology, anthropology, psychology, economy, and physics as well. All these disciplines have in common a deep interest in the investigation of the strategic behavior of individuals, even if with different cultural, psychological, behavioral approaches.

Starting from celebrated works like [27] and [3] or historical descriptions of fashion like [13], sociologists and psychologists have been trying to investigate the role of fashion and style onto the relations and the interactions that may occur among individuals. In particular, [11],[12] and [22] focus on the role of aesthetic and models on the overall phenomenon of fashion, whereas [18] describes the role of taste with a sociological approach. [14] gives a psychological insight regarding the motivations and desires caused by fashion. [1], [5], [8], [28] and [25] propose an analysis introducing the concepts of cycle and cascades, which are all key aspects of every fashionable system.

As stated above, also economists are extremely interested in fashion, since this phenomenon plays an important role in the forecast of consumer behavior. A deeper understanding of the causes and consequences of fashion cycles would lead to an important improvement in the investigation of consumer's preferences and choices. Hereby a review of the recent economic literature related to the modelling of fashion is presented. [10], [24], [15] introduce the concept of status as a key aspect of the fashion process; [21] and [23], instead, stress the role of fashion leaders. [26] tries to investigate the importance of creativity on the fashion market, whereas [17] analyzes the historical evolution of the fashion industry by means of the network relationship that occur in the fashion market. [6] provides an insight on problems linked to the modelling of fashion.

Finally papers like [2] and [4] focus on the role of fashion as a way to show ideas and opinions or to identify in some particular social group. They stress the importance of the cultural role of fashion, in groups that need to show the belonging to specific categories as American adolescences or women in the Middle East.

By starting from this synthesis of the existing literature, it is possible to understand why investigating the phenomenon of fashion requires a highly interdisciplinary attitude. Even when dealing with the economy of fashion, it is not possible disregard the psycho-sociological implications. Fashion induces very interesting examples of strategic coordination behaviors: individuals with fashionable attitude change their behavior according to other's individual choices, in order to be *à la mode*. This means that the collective behavior of a fashionable population may vary significantly depending on which type of interactions and encounters are taking place among agents. This is the main reason why from a quantitative point of view, fashion modelling seems to be so challenging.

However, there are some recent attempts to model fashion as a cultural propagation phenomena by means of a quantitative approach, considering behavioral economics, agent based models and game theory. In particular: [7] develops a simple model on fashion, based on a game theoretical approach, [9] is an attempt to describe cultural propagation phenomena, [16] proposes an agent based model regarding fashion, which combines elements of standard economics and theory of agent behavior, [19] develops a model of fashion testing different patterns of consumption, [29] and [20] develop agent based models which analyze and compare how agent consumption changes when preferences change.

In all these works, one very different from the other, three key general concepts, to be considered a paradigm for any further development, may be deduced:

- The importance of the agent status in detecting what is fashionable or not;
- The idea of consumer leaders: there must be people which influence others consumption choices;
- The idea of fashion cycles.

This present paper goes in this direction, it develops an the agent-based models taking into account the concepts investigated in the cited literature. It proposes to further investigate the role of fashion leaders and common consumers on the aggregate outcome of the system.

The contents are organized in 3 sections that follow the above introduction:

- Section 2 deals with the setting of the agent based model, and the definition of the sets of agents and the functions that identify them.
- Section 3 defines different dynamic behaviors for different types of agents.
- Section 4 deals with the numerical simulations of the models and provides comments and comparisons among different strategic behaviors, with regards to the role of specific parameters of the model.

2 The model

In this section we propose and define an agent based model which aims at investigating the mechanism which rule the phenomenon of fashion consumption. The model is characterized by a deterministic utility function which describes the level of satisfaction of the different agents and by a stochastic dynamic process which rules the choice mechanism of the agents in the considered population.

Let us consider a population composed by $N+K$ agents, each agent i is described by a uniformly distributed state w_i. Notice that the state w_i is considered an intrinsic parameter of the agent i, which does not change over time. In the more general case the state w_i may be consider a vectorial quantity, composed by different components describing different features of individuals affecting consumption choices such as age, wealth, education or social status; for simplicity we will consider the case in which w_i is scalar. This parameter may be related to the sociological concept of *taste*: as in every day life different individuals have different tastes affecting their choices regarding what to consume or not to consume, in this models agent i with state w_i will have, in the most general case, different consumption patterns than an agent j with state w_j. Notice that w_i does not depend on time t, though in a more realistic version of the model it would be possible to relax this assumption: in societies people do change state, as they grow up, change job...

Within the same population, we will consider two classes of agents:

• The *common agents* $i = 1, ..., N$, with state w_i, who decide to consume as a consequence of everyday life interactions,

• The *fashion guru agents* $k = 1, ..., K$, with state w_k, who are supposed to be trend setters having an active role in influencing the common agent consumption desire.

Fashion guru agents are supposed to be much less than common agents: $K << N$, since in general given a population of individuals, the number of trend setters or opinion leaders is limited with respect of the total size of the population. Moreover, while the common agents represent common people, fashion gurus may represent fashion stylist, or more in general, all those categories of individuals or firms that may significantly affect people consumption needs and choices: most popular teen-agers in schools, celebrities and so on. According to the different context under consideration, it may be possible to think of fashion gurus in different ways.

The model is supposed to be a Potts model, i. e. the common agent i at time t can choose among q different pattern of consumption: $\omega_i(t) = 0, 1, ..., q-1$, $\forall t$. The fashion guru agent k, instead, is characterized by a fashion consumption pattern $\omega_k(t) = 1, ..., q-1$. The pattern chosen by fashion gurus will affect the patterns chosen by common agents, as we will see in the following section. Let us suppose that $\omega_i = 0$ represents the choice of not buying, namely those agents that at time t are not consuming are identified by $\omega_i = 0$.

We can define the utility of common agents and fashion gurus, respectively, as follows:

$$U_i = \begin{cases} \sum_{k \in fashiongurus} K_{ki} \delta_{\omega_i, \omega_k} \lambda^{a_i} + \sum_{j \neq i} J_{ij} \delta_{\omega_i, \omega_j} \qquad \text{if } \omega_i \neq 0, \\ 0 \qquad \text{if } \omega_i = 0, \end{cases} \tag{1}$$

$$U_k = \frac{\sum_{i \in agents} \delta_{\omega_k, \omega_i}}{1 + \sum_{k' \neq k \in fashionguru} \delta_{\omega_k, \omega_{k'}}}, \tag{2}$$

where δ is the Kronecker symbol, $J_{ij}(\mid w_i - w_j \mid)$ and $K_{ij}(\mid w_i - w_k \mid)$ depend on the distance in state between agent i and agent j and agent i and fashion guru k, respectively. $\lambda < 1$ is a parameter describing novelty and a_i is the variable corresponding to the number of times agent i does not make a different choice. This means that not changing choice, namely becoming out of fashion, makes the utility function decrease. The common agent utility function takes into account both the interaction fashion guru-agent and the interaction agent-agent: the utility of each agent is higher if other common agents make its choice and if it makes one of the fashion gurus choices. Every time agent i makes a different choice, a_i is set to 0. Notice that the utility of fashion gurus increases if many agents follow their choices, but decreases if other fashion gurus do.

Let us define J_{ij} as

$$J_{ij} = \begin{cases} 1 \qquad \text{if } \mid w_i - w_j \mid < d, \\ 0 \qquad \text{if } \mid w_i - w_j \mid > d. \end{cases} \tag{3}$$

$\mid w_i - w_j \mid$ is a distance in state, notice that, in the case of vectorial state, this distance can be the sum of the distance in state in every component.

Following the same reasoning, let us define K_{ki} as

$$K_{ki} = \begin{cases} A & \text{if } \mid w_k - w_i \mid < d, \\ \\ 0 & \text{if } \mid w_k - w_i \mid > d. \end{cases} \tag{4}$$

where $A >> 1$ since the agent satisfaction in following the fashion guru must be higher than the one in following another common agent.

3 The dynamics

In this section we introduce the dynamics used to further describe the model: different dynamic behavior for different classes of agents will be defined. The dynamic behavior regards the different consumption choices that common agents or fashion gurus make: since the nature of these two classes of agent is different, let us suppose that common agents i may change their consumption choice every time t, whereas fashion gurus may change it every time T, with $T >> t$. It is easy to understand the reason of this assumption: while ordinary people can choose and consume whenever they want, fashion gurus (considered both as fashion industries and celebrities) cannot change so frequently, due to budget and image constraints. Fashion industries propose new fashion collection every six month and celebrities generally do not vary their look more often since they want to deliver a clear message of style, just to show some basic examples. Moreover since collections have to be designed and physically produced and advertised, a window of time is required from the old to the new collection.

As far as the common agent is concerned, let us define the following dynamics: common agent i makes a different choice at every time $t+1$ if its utility at time t is below the mean utility of common agents at time t. In this case, three different kinds of consumption behavior may occur at time $t+1$:

• ***Random behavior***: when the common agent i randomly chooses among those consumption options which he/she had not chosen at time t. Namely, if $\omega_i(t)$ is the choice of agent i at time t, at time $t+1$ the choice of agent i, $\omega_i(t+1)$, must be different from $\omega_i(t)$.

• ***Rational behavior***: when the common agent i randomly chooses among those consumption options which maximize the utility function of agent i at time t. Namely, if $\omega_i(t)$ is the choice of agent i at time t, at time $t+1$ the choice of agent i, $\omega_i(t+1)$, must maximize the utility function of i at time t.

• ***Fashion guru biased behavior***: when the common agent i randomly chooses among those options which had been taken by one of the gurus at time t. Namely, if $\omega_i(t)$ is the choice of agent i at time t, at time $t+1$ the choice of agent i, $\omega_i(t+1)$ must be the same as at least one choice made by fashion gurus, $\omega_k(t)$, at time t.

As far as the fashion guru behavior is concerned, instead, we can define more sophisticated dynamics depending on parameters regarding the utility of fashion gurus. More in details, a fashion guru changes at time T if at least $K/2$ fashion gurus had made the same choice he did at time $T-1$ or if its choice has not become popular, namely if less than N/K consumers had made his/her choice at time $t-1$. Notice that this second threshold corresponds to the fashion guru's market share. If at least one of these two conditions is satisfied, the fashion guru k at time $T+1$ may behave as follows:

• ***Random fashion guru behavior***: when the fashion guru k randomly chooses among those consumption options which he/she had not chosen at time T. Namely, if $\omega_k(T)$ is the choice of fashion guru k at time T, at time $T+1$ the choice of fashion guru k, $\omega_k(T+1)$, must be different from $\omega_k(T)$.

• ***Random different behavior***: when the fashion guru k randomly chooses among those options which had not been taken by any of the fashion gurus at time T. Namely, if $\omega_k(T)$ is the choice of fashion guru k at time T, at time $T+1$ the choice of agent k, $\omega_k(T+1)$ must be different of any choice of fashion gurus, $\omega_k(T), \forall k = 1, ..., K$, at time T.

Notice that these two different kinds of dynamics try to explain the different behavior of ordinary agents and fashion gurus: while people interact very often and it is relatively easy for them to change consumption choice, fashion gurus behave more strategically in order to be imitated by the largest number of agents.

4 Simulation results

In this section we propose simulations of the model according to different choices of parameters, in order to investigate which role these parameters play in the dynamic behavior of agents and fashion gurus. The investigation will be done for the random, rational and fashion guru biased behavior of agents and for the random behavior of fashion gurus: when fashion gurus have a random-different behavior no significant differences from the pure random behavior have been found.

In the figures, different agents and different fashion gurus are identified by different colors lines, no-choice agents by a black color line and out-of-fashion agents by a dashed line. Let us start with analyzing the different results obtained with the different behaviors of agents. The results that follow are obtained with the following choice of fix parameters: $T = 10t$, $\lambda = 0.9$, 1000 agents and $q = 0, ..., 6$ different possible choices.

4.1 Random-Random behavior

Let us remember that, according to the previous section definitions, in this case agents and fashion gurus show a totally random behavior. We assume to have connectivity, $d = 1$, namely all the agents and gurus interact with everybody.

We start describing the agents' behavior when there are 3 fashion gurus and $A = 3$, namely fashion gurus are three times more powerful in imposing their taste than common agents. As simulations show, agents behave in a rather unpredictable way, although some fashion guru patterns are recognizable in the agents' choice ones. The agent mean utility minima correspond to the time when fashion gurus make new choices: at these times the number of "out-of fashion agents", namely those agents that do not follow a trend setter proposal, represented by the dashed line in figures, is maximum. This is reasonable since at every $10T$, fashion gurus change choice. Agents need time to react to this change, changing their consumption pattern in order to be fashionable again. Notice that, after every $10T$ no-choice agents (those agents making no choice, represented by the black line) and out-of-fashion decrease regularly as to indicate that agents react properly to fashion gurus behavior. See figures 1, 2, 3, 4. As the number of gurus increases, the number of utility function minima increases too: more gurus make the agents more hesitant among all the possible options, and then less coordinates. Moreover, fashion gurus suffer from a higher market competition, that leads to less market share for each of them. See figures 5, 6, 7.

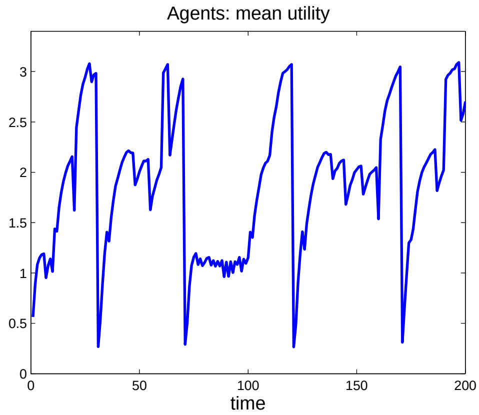

Figure 1: Agent mean utility, random case, 3 gurus

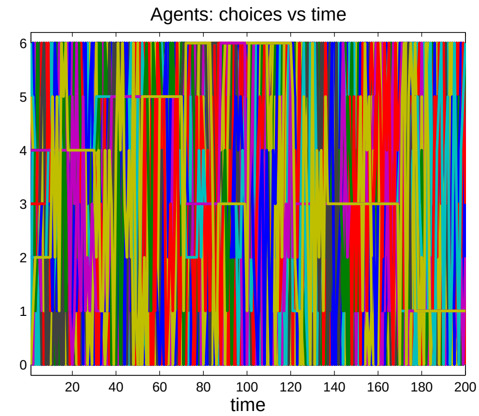

Figure 2: Agents' behavior, random case, 3 gurus

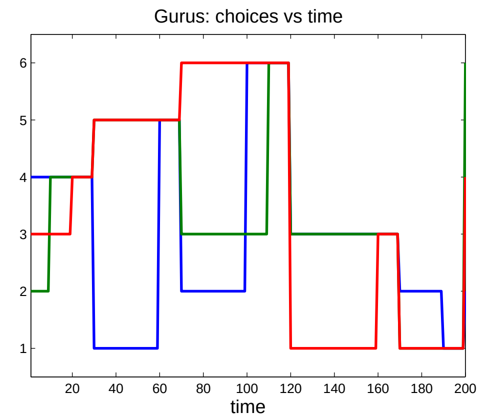

Figure 3: Fashion guru behavior, random case, 3 gurus

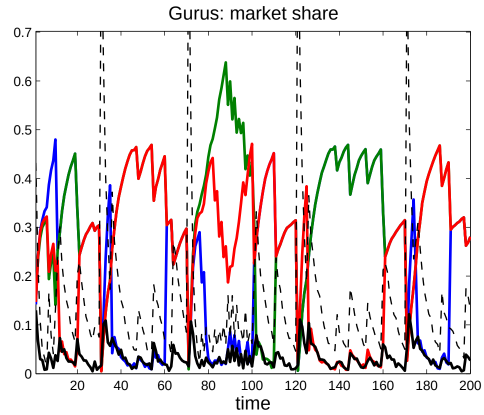

Figure 4: Fashion guru market share, random case, 3 gurus

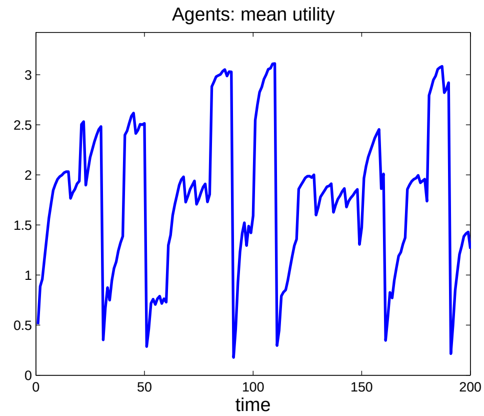

Figure 5: Agents' mean utility, random case, 7 gurus

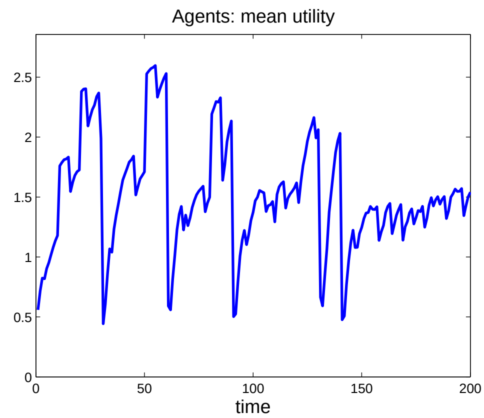

Figure 6: Agents' mean utility, random case, 30 gurus

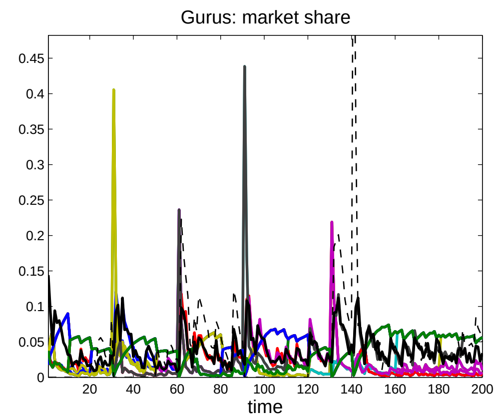

Figure 7: Fashion guru market share, random case, 30 gurus

4.2 Rational-Random Behavior

According to the previous definitions, in this case agents behave in a perfect rational way, fashion gurus, instead, show a random behavior.

As it is possible to see from figures, 8, 9, 10, 11 , agents perfectly coordinate: the shape of functions completely differ from the one of the random behavior case. Figures show some agents' hesitation when fashion gurus change choice, but soon perfectly coordinate again. Notice that the agent utility function minima correspond to the maximum numbers of out-of-fashion agents. No no choice agents are presents since agents have a perfectly rational behavior: making no choice would mean distinguish both from fashion gurus and other people, so that the resulting utility function would be low. As the number of gurus increase, we can recover a similar behavior to the agents' utility function of the random case, see figures 12, 13, 14.

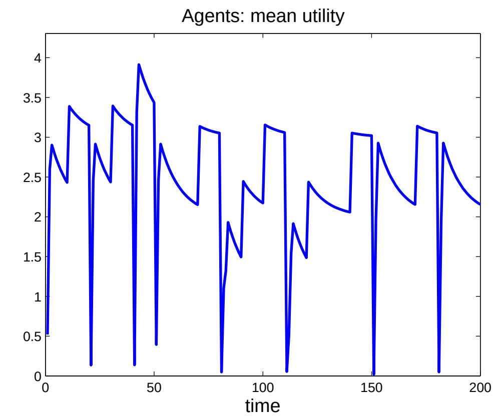

Figure 8: Agents' utility function, rational case, 3 gurus

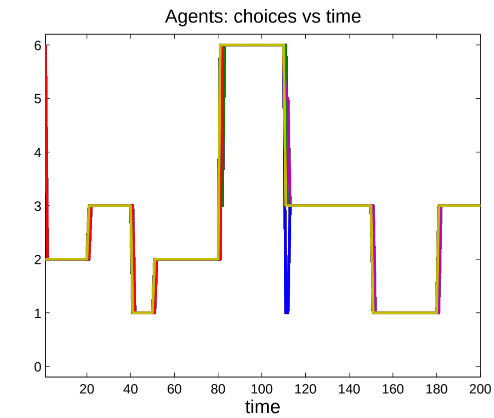

Figure 9: Agents' behavior, rational case, 3 gurus

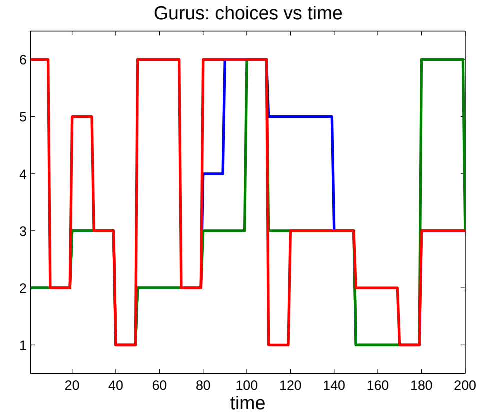

Figure 10: Fashion guru behavior, rational case, 3 gurus

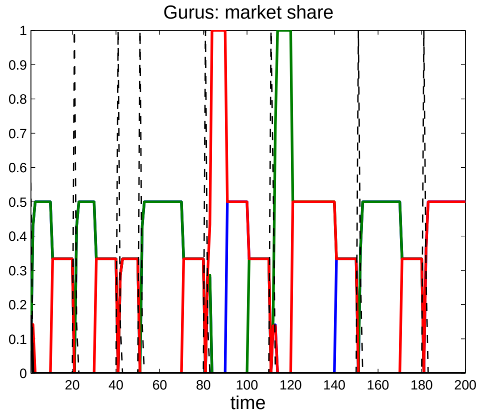

Figure 11: Fashion guru market share, rational case, 3 gurus

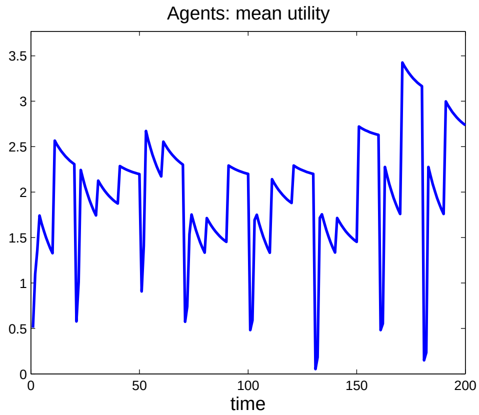

Figure 12: Agents' mean utility, rational case, 7 gurus

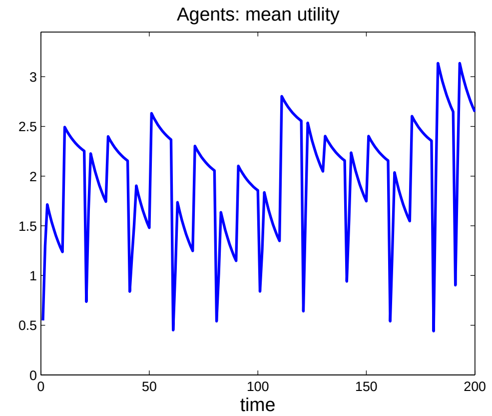

Figure 13: Agents' mean utility, rational case, 30 gurus

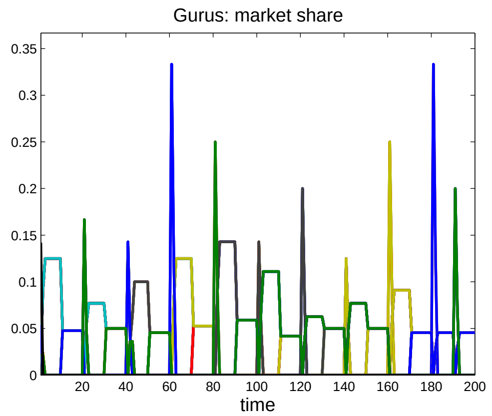

Figure 14: Fashion guru market share, rational case, 30 gurus

4.3 Guru biased-Random Behavior

According to the previous definitions, in this case, agents show a bias in favor of fashion guru choices, fashion gurus instead, show a random behavior.

In this case, out-of-fashion agents are less than in the random and rational behavior cases and no-choice agents are not present. This is an evident consequence of the bias for fashion gurus. Notice that the windows of time in which the fashion guru market share function is oscillating correspond to those windows of time when agents are not coordinated at all: the less coordination among agents, the more the oscillations. Agents' mean utility is generally higher than in the random and rational behavior cases. Agent choice remains always within the fashion guru path. As the number of fashion guru increases, the range of agent oscillations increases as well, because there are more path allowed. The oscillations of the mean utility function, instead, are more frequent but less wide since agents can easily choose a fashion guru option. As the number of fashion gurus increases the fashion gurus market share appears to be more stable. See figures 15, 16, 17, 18, 19, 20, 21.

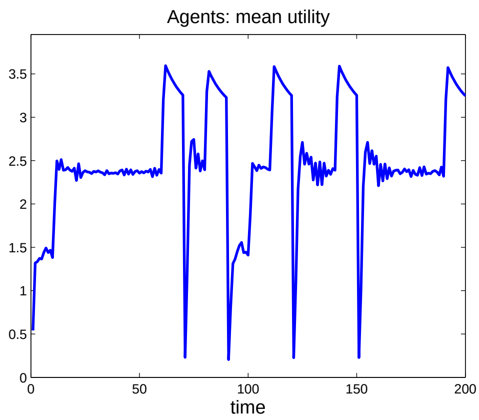

Figure 15: Agents' mean utility function,guru case, 3 gurus

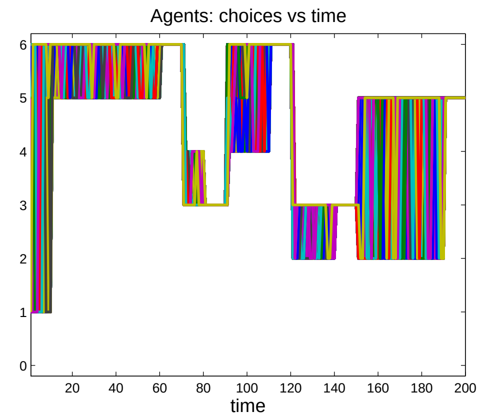

Figure 16: s' behavior, guru case, 3 gurus

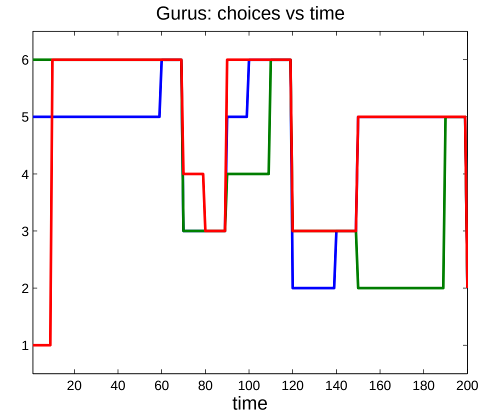

Figure 17: Fashion guru behavior, guru case, 3 gurus

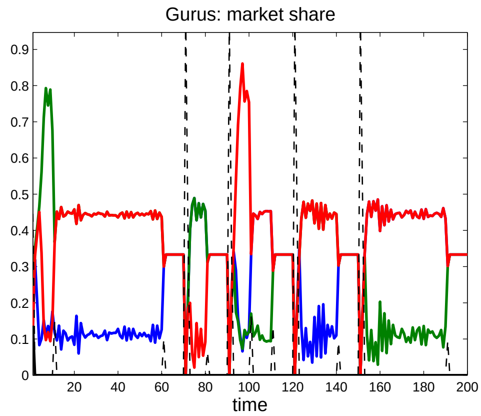

Figure 18: Fashion guru market share, guru case, 3 gurus

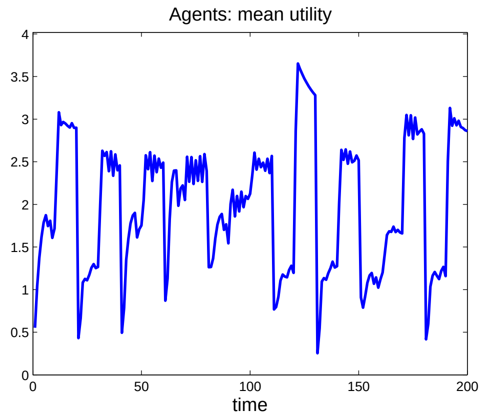

Figure 19: Agents' mean utility, guru case, 7 gurus

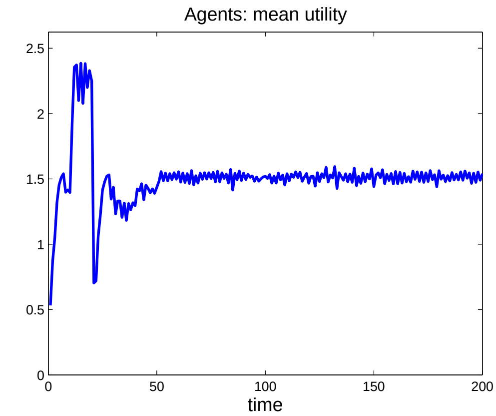

Figure 20: Agents' mean utility, guru case, 30 gurus

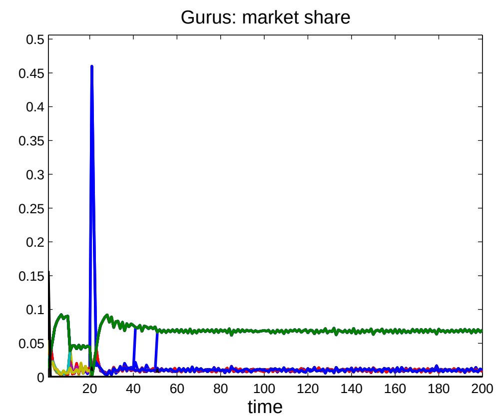

Figure 21: Fashion guru market share, guru case, 30 gurus

4.4 When A increases

An increase in the parameter A of the model means an increase in the power of fashion guru influence over common agents. As a consequence it is possible to see a very regular decrease of no choice and out-of-fashion agents (when present), whereas an increase of mean utility. As it is possible to see from the figures, agents tent to choose less options, and then oscillate less, since, as A increases, they are more attracted by fashion guru patterns. See figures 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33.

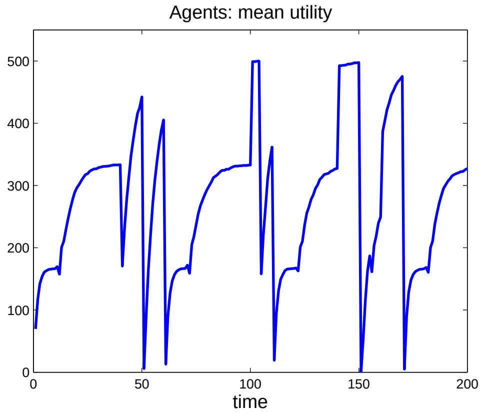

Figure 22: Agents' mean utility, random case, 3 gurus, A=500

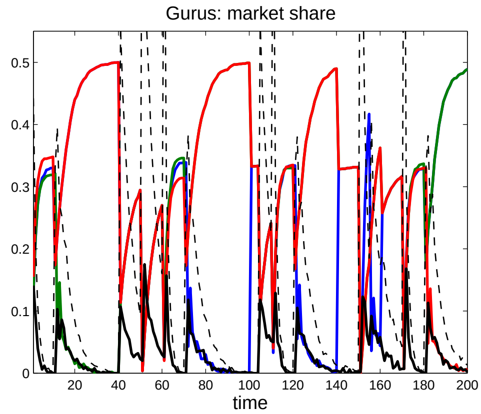

Figure 23: Fashion guru market share, random case, 3 gurus, A=500

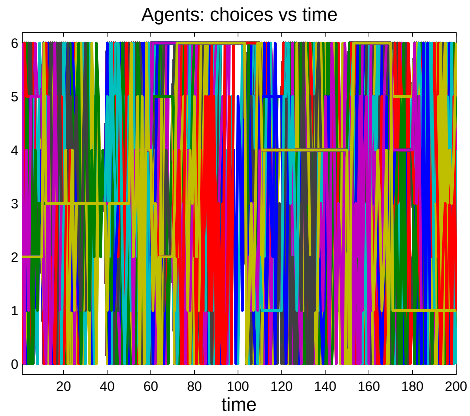

Figure 24: Agents' behavior, random case, 3 gurus, A=500

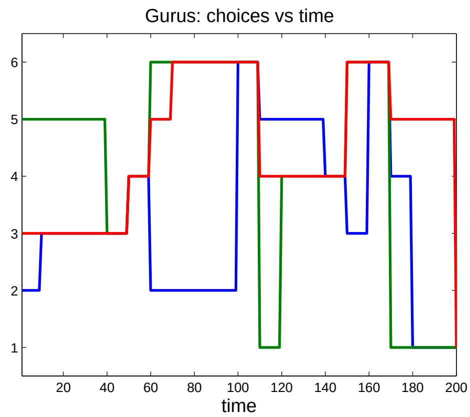

Figure 25: Fashion guru behavior, random case, 3 gurus, A=500

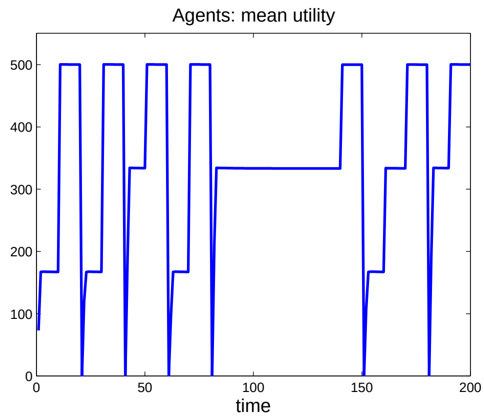

Figure 26: Agents' mean utility, rational case, 3 gurus, A=500

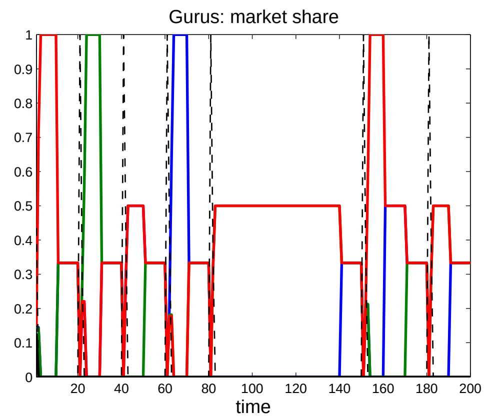

Figure 27: Fashion guru market share, rational case, 3 gurus, A=500

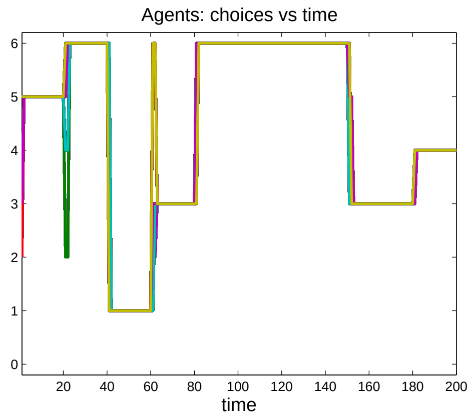

Figure 28: Agents' behavior, rational case, 30 gurus, A=500

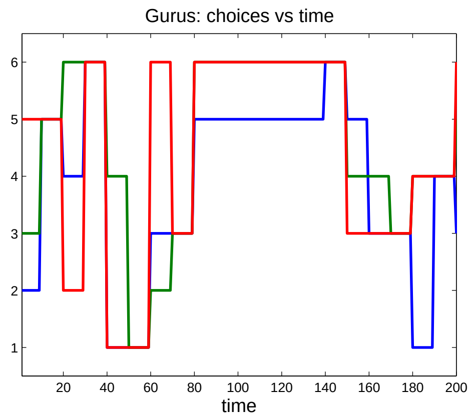

Figure 29: Fashion guru behavior, rational case, 3 gurus, A=500

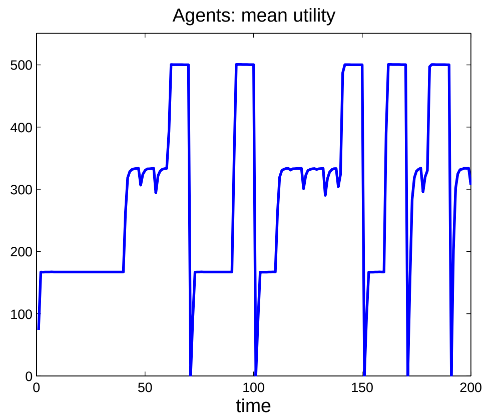

Figure 30: Agent mean utility, guru case, 3 gurus, A=500

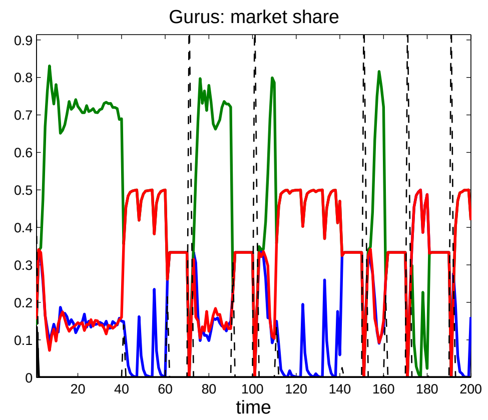

Figure 31: Fashion guru market share, guru case, 3 gurus, A=500

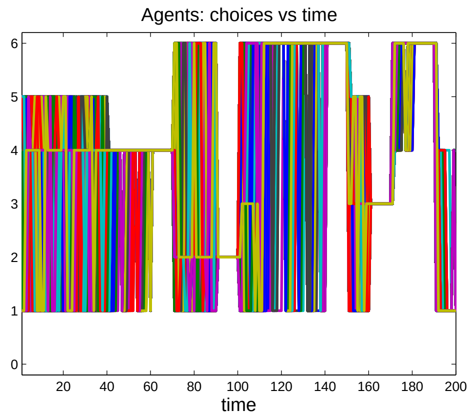

Figure 32: Agents' behavior, guru case, 30 gurus, A=500

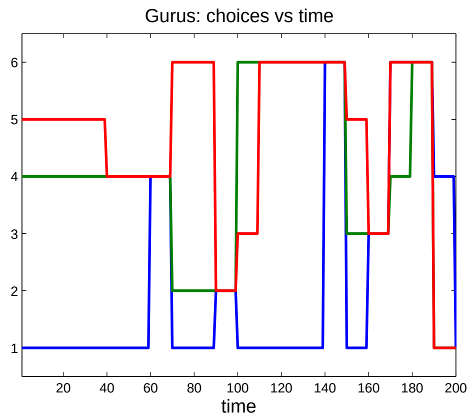

Figure 33: Fashion guru behavior, guru case, 3 gurus, A=500

4.5 When the distance decreases

Finally, let us give some comment with regards to the connectivity of the network of interactions. If the distance parameter d is less than 1, the network of interactions is reduced: agents do not interact with all the other agents of the population, but only with those agents and fashion gurus sufficiently close to their taste. In this case, results show different outcomes. Let us take into account the case of $d = 0.3$. The graphs of the agents' mean utility and fashion guru market share appear to be less regular in cycles than in the case of complete network interactions, with a significant increase of out-of-fashion agents, in the random behavior case: see figures 34, 38, 42 and 35, 39, 43. Since many agents are no more affected by fashion guru choices, they show an irregular behavior, where cycles are less evident: rational agents, for examples, coordinate less, showing a higher number of fluctuations, as it is possible to see in figure 40.

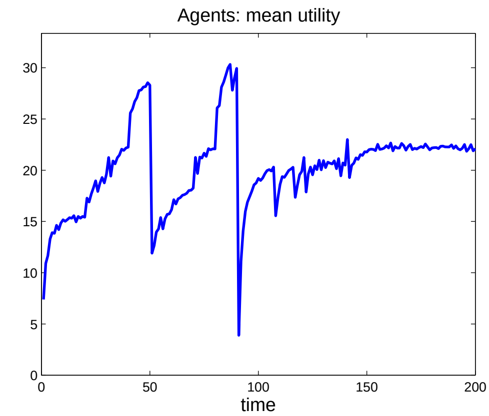

Figure 34: Agents'mean utility, random case, 15 gurus, d=0.3

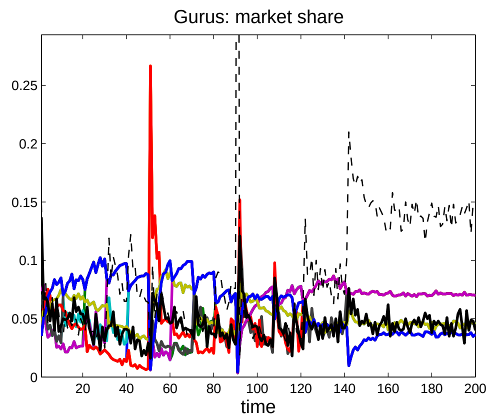

Figure 35: Fashion guru market share, random case, 15 gurus, d=0.3

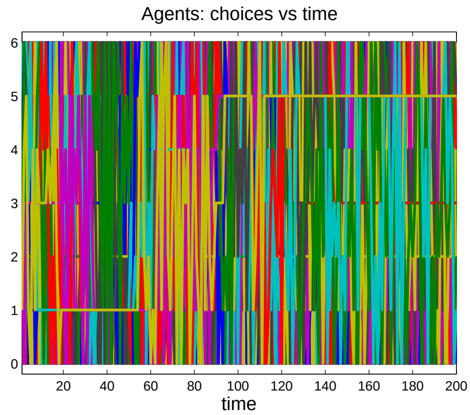

Figure 36: Agents' behavior, random case, 15 gurus, d=0.3

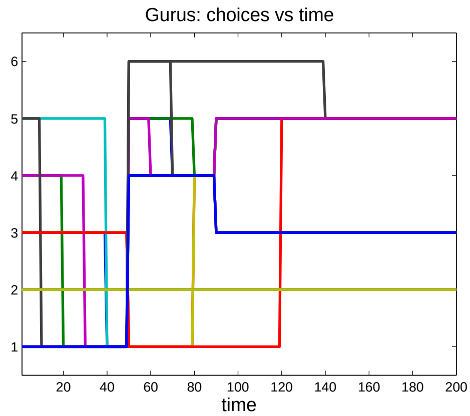

Figure 37: Fashion guru behavior, random case, 15 gurus, d=0.3

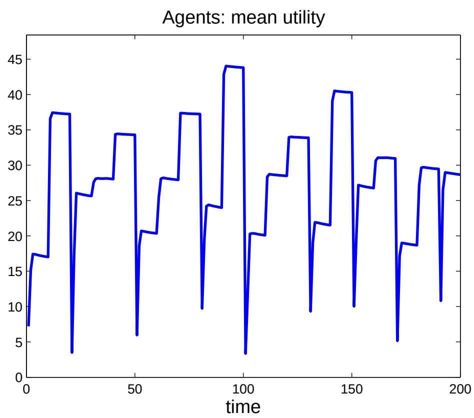

Figure 38: Agents' mean utility, rational case, 15 gurus, d=0.3

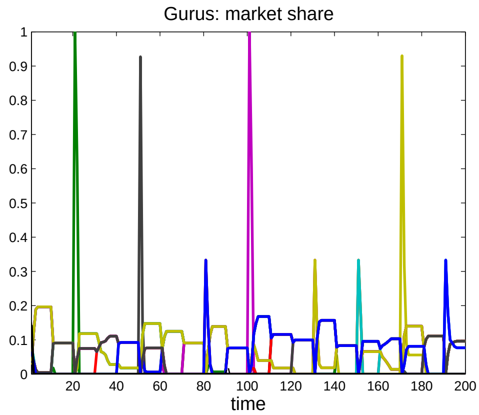

Figure 39: Fashion guru market share, rational case, 15 gurus, d=0.3

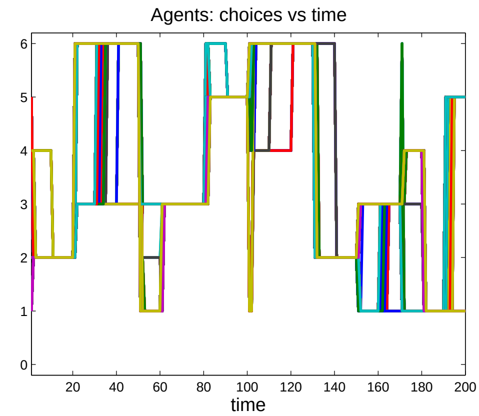

Figure 40: Agents' behavior, rational case, 15 gurus, d=0.3

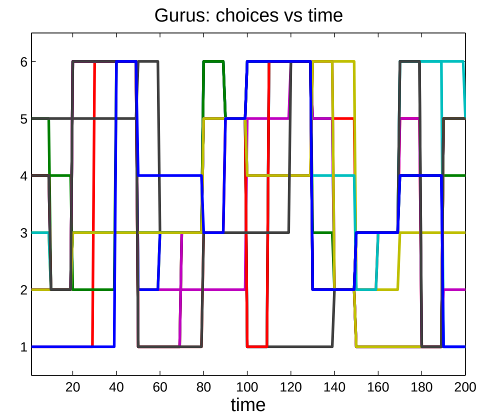

Figure 41: Fashion guru behavior, rational case, 15 gurus, d=0.3

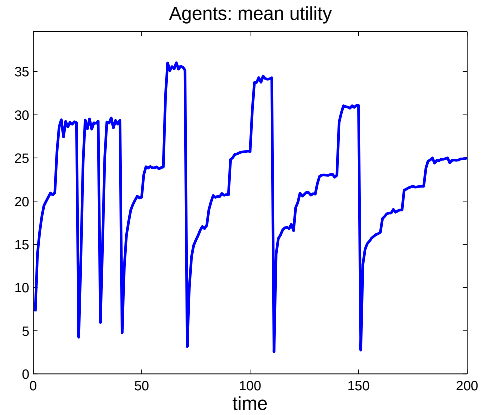

Figure 42: Agents' mean utility, guru case, 15 gurus, d=0.3

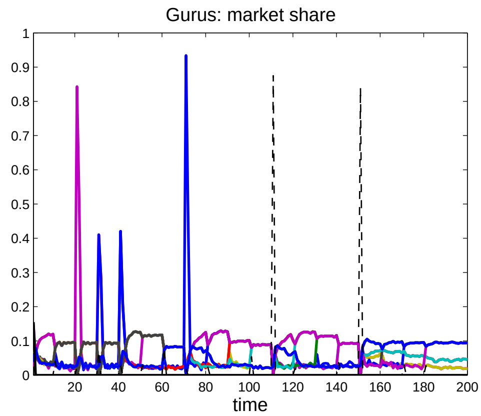

Figure 43: Fashion guru market share, guru case, 15 gurus, d=0.3

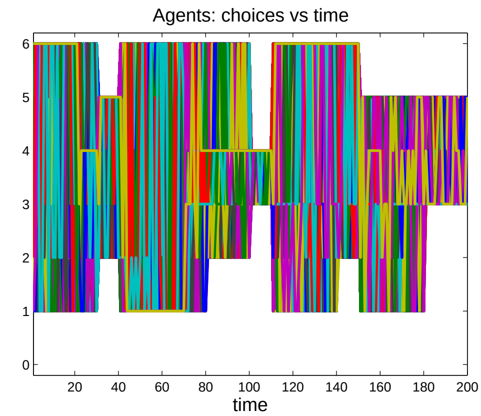

Figure 44: Agents' behavior, guru case, 15 gurus, d=0.3

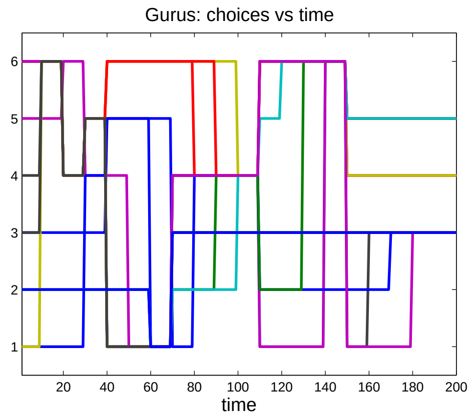

Figure 45: Fashion guru behavior, guru case, 15 gurus, d=0.3

4.6 Conclusions

This simple agent based model tries to provide a preliminary investigation on which kind of behavior and which type of parameters affect the outcomes of real fashionable behaviors. Even with a very simple model, it is possible to have completely different outcomes for a proper setting of the parameters. The model capture the relevant features of fashion behavior such as status of agents, status of trend-setters and fashion cycles. It defines a double class of agents and dynamics, in order to represent the mutual influence that fashion gurus and common agents exert one over the other.

Moreover, the three behaviors defined for the class of common agents reflect, even if in an extreme manner, behaviors of people in real life. It is possible to claim that common people generally behave like in the random case, people in fashionable contest behave like in the guru biased case and in extremely small group of people even the rational behave can be observed: let us think of the behavior of teen-agers in the same high-school where everyone can observe girls and boys dressed in exactly the same way, varying according to years and seasons.

To conduct a more precise investigation, it would be definitely interesting testing the model according to real data, coming from the fashion market. A calibration of parameters would lead to key improvements.

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