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On Logical Quantitative Methods in Politics

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Matteo Caruso

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To anyone who bets and makes a matter of life on what is not yet science, but it might be, and relies on the ideal of a fair human capital market, the only one capable of achieving the former.

To Gloria, Miranda, Clara, Leonardo, Ferdinando, Gaetano, Mario and Luigi.

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Acknowledgements

This work is the fruit of a rare passion out of place, not widespread in Italy, which I accidentally discovered while studying on a political science book in the first year of my bachelor's degree, from a simple index, the Effective Number of Parties (N) that almost all political scientists know. The Logical Quantitative Methods (LQMs) are the subsequent studies carried out by the inventor of the index N, Rein Taagepera; nevertheless, these are almost obscure and marginalized by most in academia. Since that simple index and the later discovery of the related publication on LQMs, I thought it was worth persevering even with little support after my master's degree in academia, but in any case, already impervious. I obstinately continued to insist in this direction believing that would be the only one suiting me: I always thought that was the only one useful to solve a myriad of social science problems with a more robust and different perspective so shockingly easily to my eyes, maybe just because I felt this approach belonged to my strings.

This preamble is crucial to understand my gratefulness to the first person who believed in me when nobody else would bet a cent, and who has supported me more than any other person that I have met until now. I am referring to my advisor, Prof. Alessandro Belmonte. Without him, this work would have never seen the light. His unique far-sightedness and the freedom of research he fostered, allowed me to research areas that were not close to him, then betting on me, and always giving centered comments. His support and encouragement have been defining to produce a comprehensive investigation on a broad subject that is very interdisciplinary and risky because of its experimental and odd nature, such as for the unconventional approach I adopted, when considering the most widespread approach in social science that rarely valorizes this kind of electricity. For all these things, I will sincerely ever be grateful to him for all my life.

Moreover, several years after my discovery of the Effective Number of Parties, my dream to meet Prof. Rein Taagepera in California happened. I succeeded in being a visiting scholar in Irvine, attending one of Taagepera's courses there, and finally, he accepted to be my co-advisor. All the experience there has been defining to my personal growth at 360°, I will never forget it, and I will ever benefit from it.

Other big thanks go to the external referees. To Prof. Di Clemente for his preciousness and constructive comments, and appreciate and valorized the originalities present in this work. Moreover, I am grateful to Prof. De Sio, for his comments aimed at publication, to have appreciated the density of contents present in this dissertation and the mathematical tricks used.

I thank my parents for their long-lasting encouragement to take my education to higher heights and to persevere in this endeavor.

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Vita

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Publications

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2011	"Buonanotte ai Sognatori" (Goodnight Dreamers), Preface pp.7-18, Arduino Sacco Editore, Roma. ISBN 978-88-6354-348-3.

Presentations

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2018 May 10-11th	SISE-SISP-ITANES-POPE - Fisciano (SA) – "Elezioni Politiche e Regionali 2017/8", Conference on the Italian Regional and National Elections. Paper Presenter. Title: "Italian Political Elections of 2018. The analysis with the implemented logical

	quantitative models: from the electoral system to the political party competition" (EN translation).
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Awards

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2010 March 1-4th	Winner of 2 on 2 propositions at the international forum "Democrazia 2.0", Turin. This is a biennial democracy forum Organizer by the professor and Former Chief of the Constitutional Court Gustavo Zagrebelsky.
2008 October 10th	Winner of 1St place for the contest: "Pescara the city for peace in the Mediterranean Sea". Winner of a personal computer.

Abstract

The first chapter introduces the methodology of logical quantitative models and its applications to political sciences. The second chapter explains the conversion of votes to seats. I use the law of minority attrition, expanding its form into a final model which is applicable from single member district to several electoral systems. The third chapter introduces the estimation of party seats from the previous elections using a weighted regression with independent variables jointly: 1) the product of the assembly size and the district magnitude, 2) the past values of the biggest party shares, and 3) the number of Effective parties and simply considered. Chapter four develops a probability density function with five inflection points which describes any party system. It better catches the asymmetries among the party seats distribution at nationwide level. Chapter five implements the Downsian (or positional) competition model that describes the left-right space occupied by each party through Beta functions that I have tested on the Italian elections from 1992 to 2018. Chapter six presents an in-depth qualitative analysis of the hypothesis that the more proportional an electoral system, the more the parties tend to centripetal competition, thus connecting ideological terms, effective number of parties and electoral system. In chapter seven, I suggest an alternative logical method to aggregate electoral flows, which resolves Goodman's problematics and provides a simpler solution than that of G. King. Chapter eight provides tools to more accurately calculate the optimal value of S (Taagepera and Shugart, 1989, p. 175), and unprecedently, the optimal value of other institutional variables such as: the district magnitude, the Gallagher's index of dis-representation, and the dis-representation index attributable to an electoral system (D_E), originally developed in this thesis. Chapter nine wants to determine an equilibrium between parties' and voters' "electoral utility", which is the quantity of dis-representation which benefits a group of parties and voters in the system, producing disutility for the others; this chapter enriches the law of minority attrition including thresholds and majority premiums (MJPs) and strategic vote, using a primary game theory approach and the "Maximin" Rawlsian theory (1971) as a benchmark for equality. Chapter ten provides an overview of links among the new tools and knowledge developed in this thesis, with the final aim of the normative building of an optimal electoral system, which can warrant both logical coherence and social equity as categorized by Arrow (1951).

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Chapter 1

Introduction

1.1. Overview

In recent years, political science, as well as other related social sciences, have seen an intensification of empirical works where quantitative methods are approached as a benchmark for publication. Statistical significance tends to be considered the main parameter of reliability to corroborate theoretical frameworks (Leahey, 2005), even though sometimes it produces biases used to pick certain hypothesis and reject others (Gerber - Malhotra, 2008). In particular, some social scientists select data that support their theories with little step-back to see what else the data might tell them, and what are the characteristics of the variables - therefore of the problems - taken into consideration, what are their logical – and then mathematical - limits.

In order to solve all of those intricate social science problems, the powerful methodologies and tools that I use extensively in the thesis are the *Logical Quantitative Models* (LQMs). These models employ the reading of reality throughout the logical thinking, configuring a “two leg science” (Taagepera R. , 2017, p. 7-11) which foresees that the scientific theories must consider two sides: 1) how things are, and 2) how these should be. The application of this perspective to the social and - in particular to - the political science problems, is both a methodological approach as the main core and the innovation offered by this dissertation.

LQMs are related to the fact that we cannot just use an inductivist approach to problem-solving, because the “science does not start from the observation and the induction does not exist” (Antiseri, 2007, p. 5-19), otherwise, this would mean applying a methodology without any control, in a mechanical way. Moreover, the epistemology Kuhn has referred to “normal science” when all the axioms used to solve a problem are not problematized; even though this can work to solve some well-codified exercises, it is only a mechanical way of doing science, without any kind of innovation. It is when we have an unsolved problem, or we want to try to enlarge the scope of the science that we really make the scientific revolution. (id. (p. 255-258)).

My thesis aims at linking political scientists’ analysis with logics, mathematics, physics, economics, econometrics and statistics. I agree on the Popperian ideal that says: «There are not disciplines, neither roots of the knowledge, or rather, of investigation: there are only problems and the necessity to solve them, a science like botany or chemistry (or let's say physical chemistry or electrochemistry) is, in my opinion, only an administrative unit»[1] (Popper K. (., 1994, vol. I , p. 35).

Logical models are also quantitative because it is possible to provide not only a merely directional relation but «we get much more out of a *quantitative* model […] *because a quantitative model has vastly more predictive power*» (Taagepera R. , 2015, p. 39).

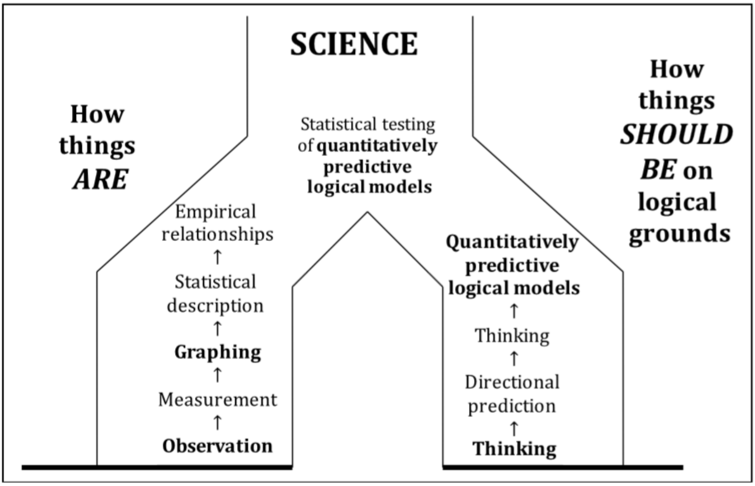


Figure 1 Science walks on two legs (Taagepera R. , 2017, p. 8).

Taagepera's approach in building LQMs applies the graphical analysis of the variables (two at a time), showing what kind of relation occurs between them, more from a perspective of the meaning of the obtained function's shape, rather than from the perspective of the statistical regression in itself as often, in social sciences, the correlation functions have a complex nature. Concerning the field of existence, sometimes we need to think inside the box (Taagepera R. , 2015, p. 66-77). Visually, this is defined by the closed part of the plane in which a relation function exists that links independent and dependent variables, such as for the model which links seats and votes shown in figure 2. Nevertheless it could also exist as a partly open box, like in the correlation between the volatility and the effective number of parties (id. (p. 96-102)), where the field of existence is unlimited – or, better, infinite – for such correlation

[1] My translation.

function. In any case, it is important to know whether some quadrants are forbidden by the possible field of existence of the function, and then being able to define them (id. (p. 118-25)).

Moreover, multiplicative relations exist over the additive ones, since the variables are frequently interlocked like in physics (Taagepera R. , 2008a, p. 52-70), in which each variable could be potentially non linearly correlated to each other, assuming a potential exponential bundle of correlation functions, also not included inside defined boundaries (id. (p. 97-106)); hence, in general, in order to build formal models, the anchor points must be considered, as well as forbidden areas (id. (p. 107-110)), and the expected shape of the correlation.

A prototypical example of LQMs, that can be useful to better understand the thesis' motivations, is given by the link between the votes and seats in a parliamentary assembly, that will be the subject of section 1.2.2, and more in-depth in chapters two and three. It concerns the strategic analysis of the alliances before – and not only ex-post - the political parties - also locally - thus producing a fragmentation (or not) of the political system. This, considered jointly with the type of electoral system used, goes to produce opposite outputs concerning the majority of seats in favor of one block or another.

This problem was first raised by Raimondo Lullo (1232-1316) and it is still present nowadays. For example, looking at the UK national election of December 2019, the sum of the forces favorable to "Remain in EU" received more votes than the "pro Brexit" forces, but because of the lack of alliances of the first block, the strategic (or swing) vote adopted by the Brexit Party in favor of Conservatives (which is a kind of alliance), and the territorial distribution of votes, gave the Conservatives 43,6% of votes but with 56,2% of seats.
Generally, elections in mature democracies present a higher probability of winning for a handful of votes and/or losing despite gaining the majority of votes. For example, in the US the Democrats lost the elections in 1876, 1888 and, most recently, in 2000 and 2016, notwithstanding receiving the majority of votes.

To transform seats into votes, Taagepera uses the function that assumes the shape of a sigmoid and is called *law of minority attrition.*

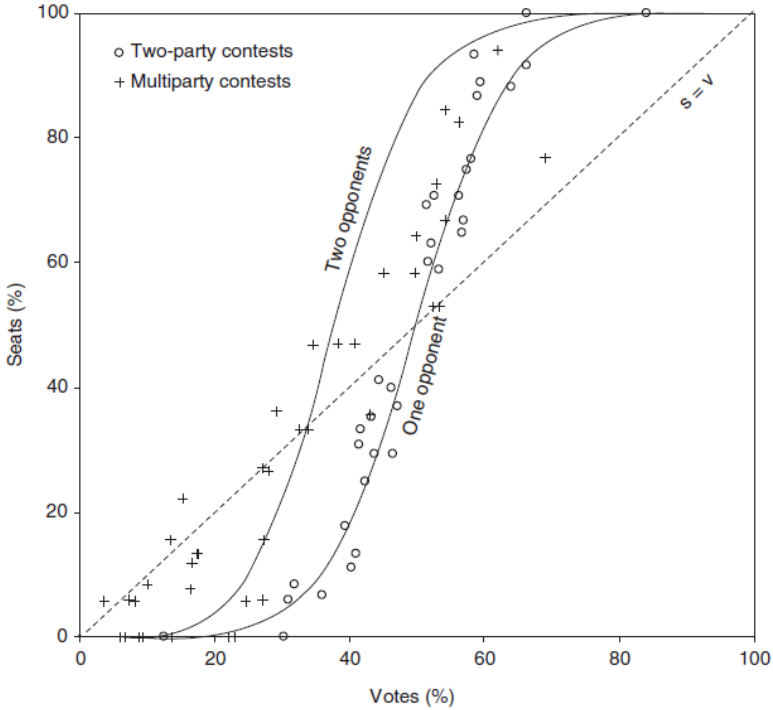


Figure 2 Distribution of Seats from Votes in FPTP (Taagepera R. , 2007a, p. 208).

The curves in Figure 2 show seats expressed as a function of votes (on a nationwide basis). This dynamic is observed in the electoral system plurality called "first past the post" (FPTP), firstly adopted in the United Kingdom (and subsequently in other countries) and still in use.

Fisichella explained: «The majority electoral system plurality in a single turn, has as a result, compared to other families of electoral systems, to maximize the percentage deviation between votes and the seats of the parties in contention» (2009, p. 283). It happens that, a party collecting little more than one-third of the votes in a district wins the seat of the district, which corresponds to 100% of the representation's share for that territory. Operatively in the FPTP, the winner (even though they do not get the absolute majority of seats) is the party able to catch the many numbers of districts and not that one taking the majority of the number of votes (on a nationwide basis). This creates the inflection point in the function describing the relation between votes and seats: the party with most votes statistically gets proportionally more seats than the votes gained, whereas the party(ies) getting fewer votes will gain proportionally fewer seats than the votes gained.

To complete the directional and quantitative shape of the curve describing the law of minority attrition, one needs to consider the "anchor points", which are the points where the function is obliged to pass, in order to fulfill some desiderata based on logical foundations as follows (Cfr. (Taagepera R. , 2008a, p. 107-111)). There are three such points in the function: 1) in (0,0) where zero seats correspond to zero votes for any party, 2) in (100,100) namely 100% of the seats correspond to 100% of votes for a party, 3) in (50,50), which is valid, assuming only two parties are competing (in the function shown in figure 2 this is named "one opponent"), in which case 50% of the votes gained correspond to 50% of seats assigned.

We can intuitively understand this function if we think that if two parties have exactly half of the votes, under conditions of equal territorial distribution among districts, then they win respectively 50% of the plurality seats apiece.

The factors which can empirically determine this disproportionality are:

1) The number of districts: the more they are, the less disproportional is the system. This is because theoretically, with only one single district, and with any relative majority, one party gets 100% of the representation; even with only two parties to compete for the district, the seat in question will reach the limit of $49,\bar{9}\%$ of dis-representation (in case the most voted party obtains 50% + 1 vote). This value of dis-representation will tend to 100% as the number of parties increase.

2) How districts are drawn on the territory and what is the territorial distribution of the votes for each respective party: the more evenly diffused the parties are on the territory, the more they benefit from the FPTP system in case they obtain the relative majority. This view is supported by the Gerrymandering principle (Fisichella, 2009, p. 276-277), which says it is possible for a party to win the relative majority of seats without gaining the relative majority of votes, thanks to the dimensional "cropping" of the (territorial) districts, even with equal dimension (number of voters). In practice, the phenomenon occurs when one party wins marginally in most of the districts (even getting the relative majority) and the same party loses strongly in the remaining

districts, thus avoiding dispersing votes.
The last methodological point to note is the logical consequence of the earlier theoretical definitions: the existence of some forbidden areas, shown in figure 3.

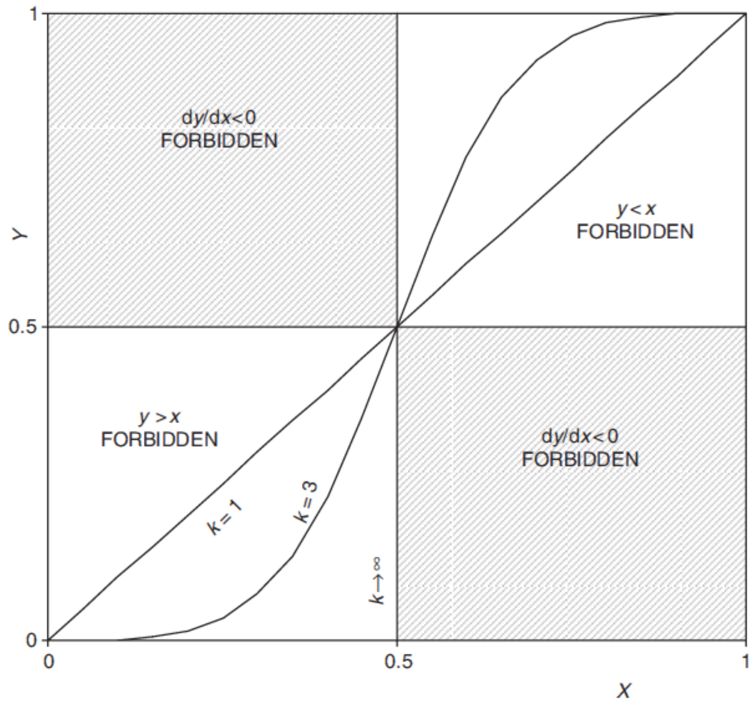


Figure 3 Forbidden regions in the law of minority attrition (Taagepera R., 2008, p.108).

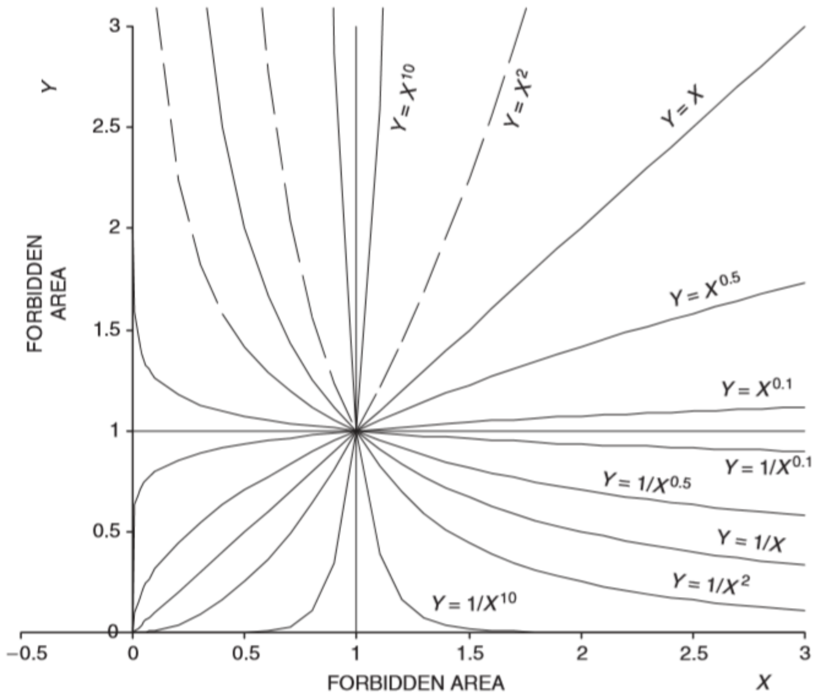
These correspond to the region of plane: 1) under the curve, from the abscissa in correspondence of the inflection point to 1; 2) over the curve, from the abscissa in correspondence of the inflection point to 0. Also, two other sectors of plane are forbidden: 3) below the abscissa in correspondence of the inflection point's abscissa to 0 such that s>v; 4) over the abscissa in correspondence of the inflection point to 1, such that s<v. (Cfr. (Taagepera R. , 2008a)).

Figure 3 also shows that LQMs can be very similar to physics, because we can have some constants beyond the variables (seats and votes) in the formal model obtained; in this case, the constant of the model is k, which determines the – marginal - degree of dis-representation, here defined from 1 to infinite. The Y discards of the function, in relation to the dashed straight-line s=v, are directly proportional to the degree of disproportionality k.

The empirical valence of the Taageperian model has a relatively low error: only 1.2% (Taagepera R. , 2007b, p. 232) obtained from the correlation between the data estimated through the LQM (law of minority attrition) and the empirical ones collected for the Caribbean islands (Nohlen, 1993), also reported in figure 2. The function tells us that the more parties compete at a nationwide level, the more the dis-representation will increase.

Even though the law of minority attrition has its own non-generalizable peculiarities, like any other complex function with inflection points, in absence of any inflection point, we can produce a generalization of LQMs. In fact, figure 4 below shows a bundle of exponential functions that pass across point (1,1) respecting the function: $y = x^k, \in \forall\, x > 1$ and $k \in \mathrm{R}^n$, however this could be easily translated, if needed, in a generalized function as follows: $y = Ax^k + c$.

Figure 4 Bundle of exponential functions that pass across point (1,1) respecting the function: $y = x^k$ and $x > 1$. *(Taagepera R. , 2008a, p. 98 (fig. 8.1)).*



In case there are more than just one independent variable, such as:

$$Y_a = \beta_1 x_1 + \cdots + \beta_p x_i x_j + \beta_n x_n + \varepsilon_a$$

and in addition, in presence of one or more of the following non-linear relations with the dependent variable, as it happens frequently in social sciences phenomena, I can apply the following notation:

$$Y_b = f_1(\vec{x}) + \cdots + f_n(\vec{x}) + \varepsilon_b \ ;$$
$$\vec{x} \in \mathrm{R}^n \ ; \ \ f_v(x_1, \dots, x_n; \overrightarrow{k_v}) = \beta_v x_v$$

$$\text{For Example:} \begin{cases} \beta_1 x_1 = f_1(\vec{x}; \overrightarrow{k_1}) \\ \beta_p x_i x_j = f_p(\vec{x}; \overrightarrow{k_p}) \end{cases}$$

Then a generalization for f_i can be:

$$f_u(\overrightarrow{A_u}; \overrightarrow{x_u}; \overrightarrow{k_u}) = A_1 x_1^{k_1} * \ldots * x_n^{k_n}$$

In LQMs A will be the constant determined by anchor points, like in physics formulations, and it will result in a unique number although it is effectively a vector because of the multiple constraints (the anchor points in fact). The parameter k will be estimated in function of the data graphing, for each variable x_i individually considered.

Below I report a series of most important physics equations (Crease, 2004) which could be an exemplum of how it could be possible to articulate the previous function $f_u(\overrightarrow{A_u}; \overrightarrow{x_u}; \overrightarrow{k_u})$.

Table 1 The 20 equations voted the most important for physics (Crease, 2004), by rank (quoted by (Taagepera R. , 2008a, p. 53 (fig. 5.1))).

Euler's equation	$e^{i\pi} = -1$
Maxwell's equations	$\nabla.\mathbf{D} = \rho$, $\nabla.\mathbf{B} = 0$, $\nabla \times \mathbf{E} = \partial\mathbf{B}/\partial t$, $\nabla \times \mathbf{H} = \partial\mathbf{D}/\partial t + \mathbf{J}$
*Newton's Second Law	$\mathbf{F} = m\mathbf{a}$
Pythagorean theorem	$a^2 + b^2 = c^2$
Schrödinger's equation	$H\Psi = E\Psi$
*Einstein's equation	$E = mc^2$
Boltzmann equation	$S = k \ln W$
One plus one	$1 + 1 = 2$
Principle of least action	$\delta S = 0$
*DeBroglie's equation	$P = h/\lambda$
Fourier transform	$F(x) = f(k)e^{2\pi kx}dk$
*Einstein's general theory of relativity	$G_{\mu\nu} = 8\pi G T_{\mu\nu}$
*Circumference of a circle	$C = 2\pi r$
Dirac equation	$i\gamma\partial\Psi = m\Psi$
Riemann zeta function	$\zeta(s) = \Pi[p^s/(p^s - 1)]$
*Hubble's Law	$v = H_0 d$
*Simplest ratio	$a/b = c/d$
*Ideal gas law	$PV = nRT$
Balmer series	$1/\lambda_n = R[1/2^2 - 1/n^2]$
*Planck's equation	$E = h\nu$

Asterisks mark the ones that follow the pattern $y = a\Pi x_i^{b_i}$.

An application done by Taagepera of this physics inspiration, is provided for the cabinet duration (see section 1.3.2).

With the law of minority attrition, Taagepera drew worldwide attention to LQMs, but this approach was already largely known thanks to his steady effort. His study on the Effective number of parties N_2 or simply N_0 (the nominal number of parties) (Laakso-Taagepera, 1979), stands as an archetype of this approach and nowadays used and known by most political scientists. This has represented a simple but strong innovation to answer the question of how to count the parties that really have a specific weight in the party system.

As the model and formalization of the law of minority attrition demonstrates, LQMs are very helpful in explaining the connections that could happen between – multiple independent and dependent - variables (Taagepera R. , 2015, p. ch. 10) on a mathematical, logical, statistical or physical basis.

The index N is the inverse of the sum of all the squares of the party's shares: by definition, N can be at least equal to one, when the party system tends toward a single party of 100% and all the others tend to 0%; N tends to infinite when the party system tends to have infinitive number of parties (having the same percentage).

Laakso and Taagepera suggested N as a variable expressing the count of parties and no other indexes, on the basis of comparative studies done about those (Taagepera - Lee Ray, 1977); (Taagepera R. , 1979 b). For example, another index, used in politics before N, has been F, made by Rae and Taylor, used to indicate the fractionalization (or fragmentation) of a party system (Rae - Taylor, 1970); (Rae, 1971). The F index is calculated as the complement to one, of the sum of the squares of the parties' shares. This index has the limitation of assuming values only from 0 to 1: Laakso and Taagepera have pointed out how F did not change appreciably for very low values, thus offering limited understanding of how many parties really have an impact on a given system.

The critical role played by parties and their `fundamental' number N in politics is well exemplified by the law of minority attrition itself. In figure 2, we see that the function can be transposed horizontally to the right and to the left of the anchor point (50,50). This is possible only in consideration of some aggregated index of the party system, which takes into account the different allocation of each party shares, thus capturing the point in which the second derivative changes (having an inflection point for $v = v_i$).

The function goes to over-represent the votes gained by a party for v> v_i, or under-represent them for v<v_i. In particular, the function has the respectively possible transpositions: 1) when N<2 the function is translated to the right; 2) for N>2, the function is translated to the left. This happens because, for exactly two parties, just one vote more from 50% could determine a big win for the party, or conversely a big defeat. When the number of parties is higher, the inflection point for which this happens is exactly equal to 1/N. This will be formalized here unprecedently, integrating the current literature.

There have been several other applications of N in political sciences. One of the most meaningful is in relation to the problem of how to measure the cabinet duration: when N goes up, the share of the cabinet's life goes down more than proportionally, also considering the specific – "political" - system's constant (Taagepera - Sikk, 2007). This will be shown in section 1.3.2. This thesis could be propaedeutic to refine this cabinet's life by adding another variable, related to the statistical distribution of the party shares, with the aim of enhancing the explicative and therefore the predictive power of the current model (Taagepera R. , 2010).

A blend of two approaches on a cabinet's duration could be proposed:

1) "duration of government II" that considers the government terminated after an election, a change of prime minister or a change of the format of the government (majority enlargement, minority government, or minority coalition) (Lijphart, 1999, p. 132-3);
2) "Average Cabinet Life I" (Lijphart, 1999, p. 132-3), proposed by Dodd (1976), which is the average duration of the executive considered even if the Prime Minister is not the same; if the parties that support him are the same, the government will be considered the same.

This thesis could be propaedeutic also to obtaining a simpler cabinet duration's model – starting from the Taagepera and Shugart formulation – that can be applied to both duration approaches, the government II and the Average Cabinet Life I. This is an important problem to investigate because it is related to the stability and the quality of democracy, which also implies, for example in political economics, that if the government's life is brief, it cannot look to make investments, but rather propend for a public expenditure finalized to a quick capitalization of electors' consensus in a brief period.

The N index has also found applications in explaining the individual level of volatility in a nationwide election in India 1998-1999 (Heath, 2005), or in explaining the interest group pluralism (Lijphart, Patterns of Democracy: Government Forms and Performance in Thirty-Six Countries, 1999, p. 183). Generally speaking, LQMs in politics have been greatly useful to estimate the biggest party in a party system, which is known only by the assembly size and the district magnitude (see section 1.2.3 and chapters two and three).

In conclusion, it is possible to define LQMs more simply as:

The methods that allow to obtain a model made of a single regressor, formed by one or many independent(s) variables - and eventually constants – jointly multiplicated. The objective of LQMs is to obtain a model much closer than and as parsimonious as possible to reality, obtained from logical assumptions and observing the empirical data plotted two by two among the variables considered.

It is testable with OLS (Ordinary Least Squares) with the aim of confirming – in the best of cases – a straight line of correlation, between the real and the predicted values obtained from the model itself. If the form of the regressor is simple, but it is suspected to be not linearly correlated, it can also be tested applying the simple logarithmic of itself, with the respective coefficient equal to the optimal exponent of correlation (Taagepera R. , 2015, p. 121-125) applicable to the whole model.

This is the reason why the foundation of this thesis starts from Taagepera's works (et al.), who uses LQMs (R. Taagepera 2005, 2008a, 2015); (Shugart - Taagepera, 2017) to make social sciences more scientific (Taagepera R. , 2008a). I start from these references, elevating the potential of this approach further. Some other researchers, such as Grofman, De Sio and Colomer, have already employed these methodologies and/or approaches, which will be shown later in detail. I am also going to explain why I follow this approach compared to others, embracing a programmatic view (not ideological and/or of academic belonging), such that: every time we modify a theory or replace it with another theory, this innovation is a step forward if and only if the modified or new theory is more efficient in solving problems than the previous doctrine (Laudan, 1979).

My thesis uses LQMs and adds some more "hard sciences" tools that Taagepera – and, more in general, the current literature – does not use, such as:

1) the differential calculus for more than two independent variables; I have unprecedently leveraged this tool to more precisely estimate the optimal assembly size considering N (with other variables and constants) as well as to optimize M and dis-representation.

2) Following from the previous point, and defining D_E as the dis-representation index attributable to an electoral system, the higher N_s and/or M*S* product, the more $D_E{}^*$ tends to 0; conversely, the lower N_s and/or M*S* product – both tending to 1 -, the more $D_E{}^*$ tends to 1, describing the correlations between N_s and $D_E{}^*$ and between M*S* and $D_E{}^*$ as branches of the hyperbola.

3) Adding the time series approach and the probability for independent events (also calculated for more than two ones) to the earlier LQMs by connecting political and institutional variables.

4) Improving the impact of electoral rules on dis-representation, furthermore calculating the features of the best electoral system for a specific party system, through optimizations, formalizing the institutional approach with the principles of game theory and specific statistical functions (such as the Eulerian's Beta for the calculus of majority premiums).

5) Taking advantage of the statistical (and logical) approach to refine all the previous formal models used. The result obtained will be a function of five inflection points using the variables:

    1) *s*, which is the independent variable represented by all possible percentages of seats that can be allotted (defined from 0 to 1 (100%));

    2) *f*, which is the dependent variable, indicating how frequent that allotment is for the respective *s*;

    3) the constant k, which will be substituted in function of N and N_0 (the nominal number of parties, with just one seat).

    For example, if there are three parties getting the same percentage of seats allotted at national level, this means that each party has $33,\bar{3}\%$ of seats, then this function must have: f equal to 0 for $0<s<33,\bar{3}\%$ and f equal to 1 for $33,\bar{3}\%<s<1$, $f=0.5$ in $s=33,\bar{3}\%$.

In brief, the research questions I am going to answer are: can we perform quantitative forecasts in political sciences about votes and seats of the parties, having some political and institutional variables? Can we measure how electoral systems impact on the party Downsian competition? How can we optimize institutional variables such as: the assembly size, the district magnitude, the dis-representation due to the electoral system (or not)? Can we derive an electoral system that satisfies *certain* desiderata?

1.2. Four Pillars

I first introduce the basic tools presented in the current literature indispensable to formulate and implement LQMs for politics. These can be summarized in four pillars, which represent the foundations of all the work. The first pillar is the *Effective Number of Parties* (N), which can calculate the number of parties that effectively count in any political system, in a simple and efficient way. The second pillar, the *law of minority attrition*, is a function at the base of the allotment of seats from votes (for the single-member district). The third pillar is represented by a simple but strong relation connecting *the political variable N with the institutional variables: assembly size S and district magnitude M, through the geometric mean*. The fourth pillar is represented by the *minimization of the conflict channels* to find the optimal value of S; this methodology is fundamental to improve the optimization of S and of the other institutional variables.

1.2.1. The effective number of parties (N)

The N_2 or simply N index was published by Laakso and Taagepera (1979), and is given by the following expression:

$$N_2 = \frac{1}{\sum_{i=1}^{N_0} p_i^2}$$

p_i represents the i-th party share that exists in the interval from 0 to 100%; N_0 is the nominal number of parties, which is the number of parties with at least one seat in the assembly. Both N_0 and N_2 can be calculated substituting p_i to votes v or seats s respectively for each i-th party (becoming s_i or v_i), which will form the base of the object of analysis. N's subscript can be equal to v or s, in case I refer to votes or seats. N can be expressed in form of concentration by the index Herfindahl-Hirschman $HH = \frac{\mathbf{1}}{N_2}$, which represents a sort of weighted average of the party shares present in each party system, with each p_i weighted respectively for itself, so that $HH = \sum_{i=1}^{n} p_i^2$. This index represents – readapting Weber (1972) - the ideal-typical party share of the party system. It is possible to also calculate the "degree of fractionalization" F of the party system, such that F= 1-HH (Rae - Taylor, 1970). Another concentration index over HH is the Gini index g,[2] generally used in economics to measure the income or tax distribution (Ardanaz - Scartascini, 2013) (Galiani - Torre - Torrens, 2016, p. 137-9). I use the Gini index here to measure how fair the distribution of the party shares is in the system, or if a single party share gets 100% (of votes or seats), this is useful to identify the asymmetries in the party distribution.

1.2.2. The law of minority attrition

The first part of the thesis is focused on how to better express the relation of seats and votes deriving from electoral systems - introduced in the overview - which Taagepera generalized in the law of minority attrition (Taagepera R. , 2007, p. 207); (2005, p. 206-211) to quantify the relationship between the seats (s), and the percentage of votes (v) – both relating to a given party -, capitalizing on the "cube law" previously applied by Kendal and Stuart to the Anglo-Saxon elections (1950).

Below is Taagepera's equation for the allocation of seats on a nationwide basis in the FPTP for one opponent (two parties which compete in the election) - as shown in figure 2:

$$s = \frac{v^n}{v^n + (1 - v)^n}$$

For two equal opponents (three parties competing in the election tending to the same size) the function becomes: $s = \frac{v^n}{v^n + 2^{1-n}(1-v)^n}$ (2007b, p. 207-209), and it is the point from which I start to interlock other variables, thus implementing the model.

The degree of disproportionality n in the formula is equal to $n = \frac{\log V}{\log S}$ (2013, p. 205) in the first formulation, with V equal to the sum of all votes expressed on a nationwide basis and S equal to the seats constituting the assembly. Knowing that E is equal to the number of districts, $E = \frac{S}{M}$, for the FPTP is valid the equation: $n = \left(\frac{\log V}{\log E}\right)$. In the proportional methods $n = \left(\frac{\log V}{\log S}\right)^{\frac{1}{M}}$ (Ibid. (p. 213)).

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[2] Given N_0, g requires to order shares p_i such that $p_i < p_{i+1}$, finally given $f_i = \mathrm{i}/\mathrm{N}_0$ indicating the maximum share for positional lag, we get: $g = \frac{2 * \sum_{i=1}^{\mathrm{N}_0}(f_i - \sum_{j=1}^{i} p_j)}{(\mathrm{N}_0 - 1)}$.

Following the graphical representation showed in figure 2, the value of 3.75 has been attributed to n for each party - using the data of the Caribbean islands (Nohlen, 1993) - because of the very small S. Here, theorized as n would be equal to 3 for the ideal-typic FPTP (1969)[3]. Then, the greater n, the greater the difference between the allocation of votes and seats, the higher the dis-representation.

1.2.3. The connection of political and institutional variables using the geometric mean.

Another simple but strong tool to interconnect political and institutional variables using LQMs is the *geometric mean*, which allows to obtain models of greater predictive power, mainly in a case where we do not have reliable and/or available data. It is used when the support (existence interval of the variables to the right side of an equation) of a relation function has equiprobable positive numbers, the median assumes major importance in respect to the mean, and therefore in absence of other information, it is reasonable to apply the geometric mean between the maximum and the minimum of its support, to find the expected value of the support (Taagepera R. , 2008a, p. 120-129).

The first application of geometric mean applied to political science came from Jean-Jacques Rousseau (1762) who suggested determining the number of members of a government by applying the geometric mean of the population. Nevertheless, the results obtained are unreasonable and leave several questions unanswered (Taagepera R. , 2015, p. 41). The first functioning political application, which connects the simple or nominal number of parties N_0 (a party sitting in the national assembly at least having one seat)[4], national assembly size S, and the district magnitude M (the number of seats allocated within the minimal territorial area), was proposed by Taagepera and Shugart (1993).

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[3] For countries with small assemblies as it is the case for small islands, the index of disproportionality n is higher than average (n = 3) and is between 3.5 and 4, the values for the Caribbean were analyzed in detail by Nohlen (1993), who reconfirms this.

[4] Taagepera in his early publications uses the notation "n" for the nominal number of parties N_0 (Taagepera R. , 1999).

Another advance is the link between the political variables – N_0 and N_2 – with the independent "institutional" variables[5], based on logical-statistical assumptions incardinated by Taagepera[6]. The first step is given by the relation $(M * S)^k = \mathrm{N}_0$,[7] where the number of parties -considered in various ways – is determined by the concomitant presence of S and M because of the Colomer's micromega rule, for which:

«Large assemblies, large electoral district magnitudes, and List PR allocation formulas with a large quota or large gaps between successive divisors—all these enhance openings for small parties. Conversely, it would seem to be in the interest of large parties to keep the competition out by having small assemblies, small district magnitudes, and small quotas or small gaps between divisors. While such knowledge has diffusely been around for some time, Colomer (2004, p. 3) compresses it in a felicitous "micromega rule": The small prefer the large, and the large prefer the small.»[8]

Taagepera has introduced the geometric mean in politics starting from the concrete problem of the Netherland assembly, composed of 100 seats - from 1918 to 1952 – allocated in a single nationwide district, using a proportional method of election (PR). How to estimate the number of parties with a presence in parliament also for just one seat? In this specific case, the result is simply equal to $\mathrm{N}_0 = \sqrt{1 * S} = \sqrt{S}$, because the arena in which the competition happens is just that of the assembly.

Nevertheless, generalizing the concept, if the district was not national, there would be a double arena, one in the districts and another in the assembly. Hence, the minimum value of the geometric mean is represented by both the district and the assembly dynamics (as before). Then the values of the geometric mean used to estimate N_0 could vary from 1 to its maximum value S, and from 1 to its minimum value M, thus obtaining two geometric means $\mathrm{N}_0 = \sqrt{1 * M}$ (for districts) and $\mathrm{N}_0 = \sqrt{1 * S}$ (for assembly); finally the geometric mean must be reapplied as follows: $\mathrm{N}_0 = \sqrt{\sqrt{M} * \sqrt{S}}$, hence generalizing:

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[5] So called by Taagepera (2007b).
[6] Idib.
[7] Cf. (Taagepera - Shugart R. e., 1993).
[8] Quoted by Taagepera (2007b, p. 84).

$$N_0 = (M * S)^{\frac{1}{4}}$$

This formula has also been positively empirically tested (Taagepera R. , 1998; 2002); (Shugart - Taagepera, 2017, p. 101-13).

A successive advancement suggested by Taagepera is represented by the resolution of the problem of how to estimate the most populated State in USA, knowing only the whole USA population and the number of states. Mirroring the same logic, in politics this can translate into: how to estimate the biggest party in a party system knowing only the assembly size and the district magnitude? The problem can be solved by applying the geometric mean between 1) the minimum possible value that the biggest country/party share can assume, which is at least equal to the arithmetic average, and 2) the whole value of the system. (Taagepera - Shugart R. -M., 1993); (Taagepera R. , 1998); (Taagepera R. , 1999).

Taagepera obtains the largest share $s_1 = \frac{1}{{N_0}^{0.5}}$ (Taagepera R. , 2015, p. 36-8), implying by definition that N cannot be logically less than $\frac{1}{s_1}$ which is defined equal to N_∞, because it corresponds to the maximum ponderation of the biggest party share s_1 in an imaginary weighted mean of s_I ,[9] resulting in $N_\infty < N_2 < N_0$. He then obtains N in function of the MS product, substituting the last inequality by the MS product and obtaining: $N_0 = (M * S)^{\frac{1}{4}}$ and $N_\infty = (M * S)^{\frac{1}{8}}$, and applying the geometric mean between N_∞ and N_0, it results that $N_2 = (M * S)^{\frac{3}{16}} \cong (M * S)^{\frac{1}{6}}$, the exponent is rounded for simplicity, but also because it is reasonable in consideration of the empirical test results obtained[10].
Hence finally, consolidating all concepts expressed so far, Taagepera formulated the following "mother" relation:

$${N_\infty}^4 = {N_2}^3 = {N_0}^2 = (M * S)^{\frac{1}{2}}$$ [11]

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[9] Being the generation function of the effective number of parties equal to $N_a = [\sum(s_i)^a]^{1/(1-a)}$ (Laakso-Taagepera, 1979, p. 6-7) , its maximum occurs for a tending to ∞.
[10] (Taagepera R. , 2007b, p. 153) tested on 25 country data by Liparth.
[11] For a wide discussion see (Taagepera R. , 2007b, p. 97;154-156;226).

1.2.4. The minimization of conflict channels

The last pillar is the optimization method that Taagepera uses, based on the differential calculus, to find the optimal S - S^*- in function of the population P. The result S^* will be the desirable number to be applied by the legislator to self-redefine the assembly size of any country having a population P.

The methodology preamble is that:

> Any social group of individuals that interact with each other – n - go to create communication or conflict channels – c - between them, even though these may only be potential; moreover «individuals who dispose of more communication channels toward others tend to have more influence and power.» (Taagepera R. , 2008a, p. 140).

To better understand this concept, I can provide a concrete basic application, assuming an equal contractual power for each member of a certain group. For *August 15th*, Clara wants to organize a lunch; if she was alone no conflict channels would be logically possible (unless she was in conflict with herself (sic!)), however, with a family reunion in mind, unfortunately some potential conflicts could arise. If the only family members are Clara and her brother, the potential conflict channel is just 1, however if there is also one parent, that number arises to 3, and if a family friend joins in, the four components go to determine 6 conflict channels, and so on. Table 2 below summarizes these values:

Table 2 Number of conflict channels in function of the number of components of a social group

Components (n)	Number of conflict channels (c)
1	0
2	1
3	3
4	6
⋮	⋮

The following formulation can formalize this progression[12]:

[12] Ibid.

$$c = \frac{n(n-1)}{2}$$

Switching this reasoning to politics, and in particular to the assembly size determination's problem, we must start with the idea that there are two conflict dimensions - of communication channels - to minimize (Taagepera R. , 2007b, p. 189-191):

1) External costs. These are represented by the sum of the communication channels - c - of each member of parliament with the voters of their own district.

2) Internal costs. These are the sum of the communication channels - c - of the assembly members in position of collective reciprocal listening (during the work of the assembly).

Hence, to find a break-even point between these two costs is necessary to minimize their sum.

The formalization of these two costs is given by[13]:

1) How many citizens each member of parliament (MP) represents is equal to $c_{1,a} = \frac{P}{S} - 1$; nevertheless the MP is in position of "reciprocal interaction", which means that the MP needs to reach for electors and electors need to reach for their own MP, therefore it is necessary to multiply this ratio by 2, obtaining $c_1 = 2\left(\frac{P}{S} - 1\right)$. Finally, for P big enough, $\lim_{P\to\infty} c_1 = \frac{2P}{S}$.

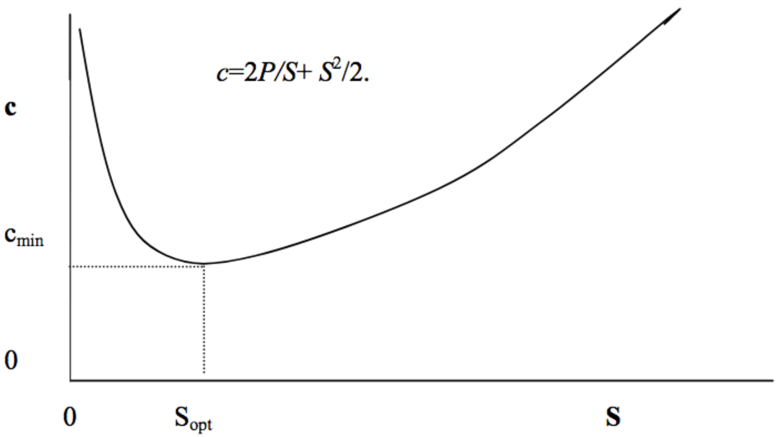
2) The communication channels of the assembly members in position of collective reciprocal listening is given by the combinatorial formula $c_2 = \frac{S(S-1)}{2}$. This is because each assembly member S should reach relations with the rest of the assembly S-1, but the simple multiplication would go to double count the communication channels, hence the product is divided by 2. Finally, for S big enough, $\lim_{S\to\infty} c_2 = \frac{S^2}{2}$.

In both points reference is done to limits for large numbers to obtain meaningful simplifications, which is why Taagepera has established P>1000 as the minimum thresold above which this is applicable

[13] (Taagepera R. , 2008a, p. 140; 2007, p. 198-99).

(Taagepera R. , 2007b, p. 198-9), where S will implicitly follow a good consistency being logically positively correlated to P. At this point it is possible to write the comprehensive formula of the conflict channels to minimize, as: $c = c_1 + c_2 = \frac{2P}{S} + \frac{S^2}{2}$; this function is drawn in figure 5 below. On the right side of the equation, the first monomial sees S at the denominator, whereas the second one has it at the numerator; this implies that: the higher S, the more the right branch of the paraboloid is determined by the communication channels among the assembly members; conversely, the lower S, the more the left branch of the paraboloid is determined by the communication channels of the members of parliament in their reciprocal interactions with their respective voters.

Figure 5 The burden of communication channels on one representative (Taagepera R. , 2015, p. 171 (fig.19.3)).



This function can be differentiated by S and it is imposed equal to 0 because in that point the function has the minimum required:

$$\frac{dc}{dS}\left(\frac{2P}{S} + \frac{S^2}{2}\right) = 0$$

In fact, its resolution identifies the point (S_{opt}, c_{min}) for which the paraboloid in figure 5 intersects the orizontal straight line $c = c_{min}$. Hence, the result of this differential calculus, will be the optimal S, that I call S^*, equal to $S^* = \sqrt[3]{2P}$.[14] Taagepera finalizes this calculus assuming that the active population corresponds more or less to P/2,[15] hence the final formulation results to be:

$$S^* = P^{\frac{1}{3}}$$

1.3. Political applications of LQMs developed in the thesis

The second part of the thesis is focused on applications also explored in Taagepera's works "Predicting Party Sizes" (2007), "Making Social Science More Scientific" (2008a) and in "Logical Models and Basic Numeracy in Social Sciences" (2015), taking an as comprehensive approach as possible regarding logical quantitative applications in political sciences, also covering positional party competition and electoral flows, in addition to cabinet duration.

1.3.1. Seats and Votes

Chapter two aims at implementing the law of minority attrition that converts votes in seats. My aim is to develop a general model obtained through an in-depth analysis of the Italian elections, chosen for the peculiar complexity of this country over time, considering: thresholds, majority premiums, the simultaneous proportional and the district's single winner or first-past-the-post system - FPTP[16] - and other fuzzy compensative mechanism of seats allotment. It potentially could allow to build a generalized model that can be applied on a cross country basis (excluding alternative vote, single transferable/non-transferable vote and their derivatives).[17]

The law of minority attrition works well for simple scenarios which are representative of FPTP, and when there is a bipartite system like that of the Caribbean Islands, but less if: 1) N is high; 2) M and/or S are low; 3) the electoral laws are complex, such as in mixed electoral systems, like in Italy, Spain, Japan and others. At which point, my hypothesis is that: if the political and institutional variables are interlocked before the law of minority attrition is introduced, I can apply the final model that works with any electoral system and functioning even better in the pure FPTP.

[14] Cfr. (Taagepera R. , 2015, p. 170-2).
[15] Cfr. (Taagepera R. , 2007b, p. 199).
[16] It is the electoral system in which M=1, like in the UK.
[17] PBV, AV, STV, SNTV, MNTV, MMM, MMP, see Shugart and Taagepera for this classification (2017, 31-60).

It is indispensable to have a powerful model which interconnects votes, seats and other possible political variables, in order to study what in economics is called comparative statics[18] (Hicks, 1939); (Samuelson, 1947). In this case the final equilibrium is the product, not of exogenous variables, but of a multivariate interlocked system of causality, which better allows to analyze and project all possible scenarios, for example in making more efficient exit polls, electoral engineering and political strategies. Actual knowledge about these tools among the political scientists, besides Taagepera, is limited to the simple directional relations, and not quantitative ones (Fisichella, 2009, p. 263-288) (Grofman, 2004) (Sartori, 2003; 1987) (Colomer J. , 2004; 2005).

To test this scenario, I enriched the law of minority attrition, starting from Taagepera's generalization of this for opponents 1 and 2 (Taagepera R. , p. 207-9)), and introducing $\mathrm{N_v}$ at the denominator in substitution of the numeric constant. Knowing that the parameter n is defined in [1,∞], following the graphical representation of seats in function of votes, applied to each party considered in several countries, the value of 3.75 is attributed to n, using the data of the Caribbean islands (Nohlen, 1993) because of the really small S as used by Theil (1969), although he states that the plurality (FPTP) voting systems will give an average value of n equal to 3[19]. To guarantee this wider application, I propose to adopt a substitution of the index n, obtaining n_1, by means of the other institutional variables P, S, M, N, E. I consider the ideal-typical value that n must assume for the FPTP, which is 3, and build the following logical-deductive assumptions:

1) where a sub-representation exists such that $P > S^3$, then the dis-representation index n will increase; inversely, for $P < S^3$ I expect a decrease of n down to 1 (for $S \to \infty, and\ P \ll S$);

2) the higher the average territorial concentration of the parties on the territory G_d, the lower n (for $G_d = 1 \Rightarrow n = 1$);

[18] In which two different equilibrium's states are caused by a variation of an exogenous variable. Unprecedently graphically shown by Fleeming Jenkin (1870).

[19] For countries with small assemblies as is the case of small islands, the index of disproportionality n is higher than average (n = 3) and is between 3.5 and 4, the values for the Caribbean were analyzed in detail by Nohlen (1993), reconfirms this.

3) considering the limit case in which N_2 tended to the anchor point 1, even the remaining party shares on the territory, although concentrated, would see a reduced probability of being able to win even one district because the presence of the bigger party would vandalize it, then $n \to \infty$. On the other side, if N_2 tended to infinity, the competitive numeric advantage for every party would be null even in the case of uninominal districts. The nullification happens because the winning probability in each district is almost identical for each party, as the victory will be determined for someone randomly thanks to a handful of votes for each district, then $n \to 1$;

4) The first member of the exponent, $\frac{S-M}{SM-1}$, aims at assessing the dimension of districts M, thus allowing the application of the model to several values of M coexisting in the same electoral system, and their relationship with S. Through a logical study of the limit cases, it is possible to deduce that for S = M there is a single nationwide district, in which the dis-proportional effect given by the mechanism of the districts would be null. On the opposite extreme, for M = 1 in presence of a pure plurality, the dynamics of the districts would fully exert their effect.

5) Defining the relation $E = \frac{S}{M}$, knowing that $1 < E < P$, and knowing that higher E implies small districts, the higher E, the lower n, then the smaller the districts, the more n tends to 1. Hence, I have defined the square root of P, applying the geometric mean of the previous domain [1, P];

Then, this formula capitalizes on the impact of the institutional variables determining a new n, on a logical quantitative base.

If there is a national threshold of representation T [0,1], it must be taken into consideration[20] through some further manipulation of n_1, obtaining a final n, equal to n_2, such that the higher T, the more n tends to ∞; conversely, the lower T, the more n_2 tends to n_1.

[20] For its maximum value between that provided by the electoral law and due to M and S (Taagepera R. , 2007b, p. 248-9).

This general model n_2 is further modified in case the specific electoral laws in mixed electoral systems present other complications, such as the majority premium and/or as happened in Italy in relation to *scorporo* – a corrective variable used from 1994 to 2001 - which consists in subtracting votes to the winning parties in the FPTP seats allotment (equal to 3/4 of S) to benefit the other lists for the proportional section (equal to 1/4 of S).

The final model obtained allows the evaluation of the impact of each political and institutional variable on disproportionality between votes and seats, aggregating their impact on representation exactly, identifying if the overall electoral mechanic is mostly proportional or majoritarian, simplifying then the effect of complex and specific electoral legislations. All these considerations represent the added value of the new disproportional index n_2, which goes to complement n.[21]

Concerning the conversion of votes in seats for the in-depth analysis of the Italian case, the elections happened from 1992 to 2018 are taken into consideration, in a postdictive analysis, configuring four different mixed electoral systems[22], and obtaining a dataset of 354 cases (data of the Interior ministry). The dataset used for the cross-country analysis is that of Struthers - Li - Shugart (2018), also used in Taagepera and Shugart (2017) on a nationwide basis - concerning countries –, composed of 974 elections, after cleaning it[23]. The same dataset is used to test almost all parts of the thesis, if not indicated differently.
The results of the model tested for Italian elections has achieved an R^2adj=97.5% between the actual and estimated seats from votes.

Chapter three introduces the prediction of party seats from the previous elections, specularly to s_1. The current literature provides purely postdictive tools given by the rules $s_1 = N_2{}^{-3/4} = N_0{}^{-1/2} = (MS)^{-1/8}$ obtainable from the four pillars[24], but here I suggest not to consider only a simple geometric mean between the previous formulations to obtain a prediction, for two reasons:

1) the presence of double (but complementary) heteroskedasticity – this means that plotting the abovementioned variables, I can observe that: the higher

[21] Cfr. diffusely (Taagepera R. , 2007b, p. 204-223).
[22] Covering the PR, MMM and MMP types (having the FPTP, the uses of T, s and the MJP).
[23] Respecting the law before introduced $N_2 \leq N_0$, because of 10 cases found for which this does not occur, it is likely given by the different sources of data collecting, as hypothesized by Taagepera in private correspondence (2019).
[24] See also (Shugart - Taagepera, 2017, p. 106-7).

the past value of s_1 - which is $s_{1,t-1}$ - for forecast s_1, the higher its error; inversely, the lower the MS product, the higher the forecast precision of s_1 in correspondence of higher values of s_1;

2) to warrant the stability of the variables over time.

I have therefore built a *hybrid institutional model*, in which N is expressed in function of M and S, and the biggest party in terms of seats and votes s_1 and v_1, in an interlocked model which weights the degree of heteroskedasticity of each one. Moreover, I have leveraged the classic tools of time series (Lutkepohl H. Kratzig M., 2004), using the past values of s_1 and v_1 to predict the current ones, because this methodology allows the application of a specific solution for politics to the problem of heteroskedasticity, in a simpler and asymptotically more efficient way compared to other methods generally used in econometrics (Arellano - Bond, 1991); moreover this allows to "institutionalize" political variables such as s_1, creating much stabler variables that are able to increase their explicative (and predictive) power.

The cross country analysis, s_1, N_{s0} *and* N_{s2} in t-1 with the MS product, predicts s_1 in t with R^2adj=73.3%, producing an improvement of the R^2adj by 29.3 percentage points, if the purely postdictive s_1 expressed in function of N_{s0} *and* N_{s2} is considered respect s_1 obtained from N_{s0} *and* N_{s2} in (t-1). Thus, the powerful result is that the predictive values are stronger than the simpler postdictive models available. As introduced in the previous paragraph, this allows to test the effectiveness of much stabler political variables from the past time.

I then move to the second part of the thesis, to expand the connections between political and institutional variables, beyond the simple problem of votes and seats, to explore other improvable political matters.

1.3.2. Cabinet duration

The answer to the question introduced in the overview regarding the cabinet duration's formalization is given by the differential calculus, applied by Taagepera, translating the reasoning done by Kochen and Deutsch (1969) to find the optimal warehouses for a firm that serves a region. Unprecedently in politics, this logic allows to obtain an optimal value of P in function of S, considering the minimization of the conflict channels inside the assembly, and those of each representative with their electors (Taagepera and Shugart, 1989, p. 175). Using the previous calculus of the conflict channels inside a party system, which can be rounded to the square of the (effective) actors of that system, Taagepera obtained N^2 (Taagepera R. , 2015, p. 49-55). Applying a physics approach, he interlocks this entity inversely proportionally to the cabinet duration, taking into consideration the "political" system's constant k, retracing the structure of the universal law of gravitation of the planets (Taagepera R. , 2008a, p. 54).

The first approach to estimating cabinet life duration comes from Taylor and Herman in 1971, followed by that of Strom in 1985. This approach is realized "through variables or attributes" that the government exhibits at the time of its formation (Laver M. - Shepsle K., Jan. 1998, p. 30). In 1984, Browne et al. identified the concept of Hazard Rate as the percentage risk of a government crisis, independently from those attributes before-mentioned, as they are measured at the beginning of the legislation; thus, as Laver hypothesized, «the opportunity costs of losing power positions decrease as legislation proceeds»[25]. This factor accounts for 20-30% of the variations of government or coalitions duration (Browne et al. (1986, p. 630)).

The hazard rate approach has been adopted in different ways by King et al. (1994 , p. 190-200; 1990 (Aug), p. 848-856, 860), Warwick and Easton (1992, p. 130-140); (Warwick, 1994, p. 94-114) proposing the "cubic hazard rate". Box-Steffensmeier and Jones (March 2004), proposed a unified model through the optimization of a generalized Gamma distribution (p.41). Finally Bernhard and Leblang (2006, p. 111-4) proposed a discreet hazard model with a probit specification.

All these approaches are referred to in the definition of "duration of government II" (introduced above).

Taagepera and Shugart (1989, p. 99-101) have instead introduced the approach linked to the definition established by Dodd (1976), and which Lijphart calls "Average Cabinet Life I" (1999, p. 132-3) (before described). This represents a great innovation, because it simplifies the model and increases the R-squared between the cabinet duration and the right-hand side of the correlation function, even though this creates a loss of information about the peculiarities of each country's political system for a specific legislature.

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[25] Laver (2003, 36), this derives from claiming that there is a formal stochastic reason and a formal intuition (Idib.).

Taagepera's actual cabinet life duration formula is $\mathrm{C} = \frac{\mathrm{k}}{\mathrm{N_s}^2}$, with k empirically determined k=42 (Taagepera R. a., 2007, p. 168-9); it presents a good $\mathrm{R}^2 = 0.77$ (Id. (2007, p. 168-70)); (Taagepera - Sikk, 2007).

The cabinet duration is an interesting example of application of a logical quantitative methodology to political sciences. In fact, as introduced in section 1.1 - in table 1 – and as introduced by Taagepera (Taagepera - Shugart R.-M. , 1989, p. 99-101), the cabinet duration formula is inspired by the formula of planet gravitation $G = k * \frac{m_1 * m_2}{r^2}$, where in this case, the generalization of $f_u(\overrightarrow{A_u}; \overrightarrow{x_u}; \overrightarrow{k_u})$ that I introduced before becomes: $C(N_2; k) = \frac{k}{{N_2}^2}$, and where the gravitational law is $F(m_1, m_2, r; k) = k * \frac{m_1 * m_2}{r^2}$.

An enormous potential arising from the application of LQMs, could be to obtain a model which blends cabinet life I and II. As Laver claims (2003), cabinet life duration is a problem that still maintains strong academic relevance (Id., p.38).

1.3.3. Spatial Electoral Competition

In consideration of the informative value of the self-collocation of electors in the ideological continuum left-right, each party will be assigned indexes of ideological positioning, placed on a left-right ideological continuum.

Chapter five introduces a revised Downsian competition, better quantifying and simplifying the ideological left-right space, in which the positional competition among parties happens. The operative tools used are Beta functions, here unparallely introduced, which are going to improve the most recent approaches (Adams - Merrill III - Grofman J. - S.-B., 2001, p. 15-51) [26]. Thanks to the post-electoral survey data from 1994-2013 recorded in the ITANES database (data request, 2018), I used the self-collocation of electors to identify the positional party competition and grouping the values for each elector per party, thus making it possible to draw the probability density function for each party concerning its left-right continuum location. The positional party competition is represented unprecedently by a Beta function, or

[26] mainly concerning an equilibrium model of party competition.

equivalently and if preferred, it can be shown in the form of cumulative functions capitalizing the logic of the first order stochasticity (Mas-Colell - Whinston - Green, 1995, p. 194-197). The data for 1992 and 2018 are obtained thanks to a new logical method exhibited in chapter seven.

Chapter six presents an in-depth qualitative analysis of the reflection done by Bernard Grofman in the article "Downs and two-Party convergence" (2004), in which the original assumptions of the Downsian theory (Downs, 1957) of parties' ideological convergence have been more precisely defined with regards to concrete political and institutional scenarios. For this purpose, I want to blend the following studies: 1) how the Downsian convergence would vary in function of electoral systems (Grofman, 2004, p. 26, 31), 2) the number of parties which compete in the election Ib. (p. 26-8), 3) consider the issues concerning the positional, non-positional and majoritarian competition (De Sio, 2011), which unparallelly I will apply to each party.

The resulting innovative logical-qualitative model links institutional variables to the parties and positional ones: modifying one will change all the others.

The aim is to use this model in comparative statics – as introduced before – where the final equilibrium is the product of the interlocked multivariate system of causality, allowing to: 1) correlate how a specific electoral system can modify the party ideological positioning on the left-right continuum, 2) know how many parties would be in the political space, and their location, with no disproportionality due to electoral systems, 3) correlate the countervailing effect of the electoral system in charge, introduced only theoretically by Sartori (2003, p. 61-2).

As introduced before, in presence of an "*Ideal-typical*"[27] plurality or proportional system[28], my theory can overcome the limits of the pure FPTP and proportional electoral systems, as it is able to also catch the shades amongst them. A pillar theory underlying my model is that the more proportional an electoral system, the more the parties tend to a centripetal competition (Sartori, 2003, p. 60-3), implying that minor parties tend to assume extreme ideological positions to be more visible. This can also be transferred into ideological terms, since the growth-survival of some parties find a fertile environment in a purely

[27] referring to the Weberian concept of the ideal-types (Weber, 1972).
[28] as well explained by Sartori with regards to electoral mechanics in "pure" plurality systems (2003, p. 57-64), the degree of disproportionality n (Taagepera R. , 2007b, p. 204-207) is quantitatively different, even though formally all districts present the same electoral system with a single winner.

proportional electoral system (as there are fewer barriers to entry) therefore parties should logically occupy all the ideological vacuums available, probabilistically smoothly spreading on the left-right positional continuum; conversely, a robust majoritarian electoral system will tend to produce a bipolar competition. The pillars of political science supporting these mechanics are: the laws of Duverger (1951; 1954, pp. pp. 247, 269; 1955, p. p. 113), the party competition of Downs (1957), and relative upgrades offered by Rae (1971, p. 95), Riker (1982, p. 760) and Sartori (2003).

To verify the relation between the final disproportionality n_2 – as introduced above - and party competition dynamics, I have collated the data from the in-depth qualitative analysis in the period analyzed for Italy's seats from votes (from 1992 to 2018) and the positional party location from the previous chapter in the same period. As mentioned, I will obtain the data for 1992 and 2018 using a new logical method exhibited in chapter seven.

The final qualitative models suggest a tri-dimensional relation between n, the average weighted positional distance px, and N; and a correlation between px and the Effective number of parties (weighted on the electoral system). A certain change of n_2 would have an impact of: mainly due to positional party competition (conditional to the non-positional one), secondly from the degree of bi-polarization of the party system, and lastly from the weighted ideological distance px.

In chapter seven, I suggest an alternative method to aggregate electoral flows.

Goodman (1953) was the first to formalize a method to estimate the "swing votes", which is the number of voters moving from one election to another from one party to another, using territorial sub-units. Unfortunately, this method can produce either some negative coefficients or an unreasonable sum (greater than 1), or both, because the votes received by each party are at least 0 and at most equal to 1 (100%). Other methods (King, 1997; King - Rosen - Tanner, 1999, 1); (De Sio, 2009,1), may solve this problem, however they are very complex to use and need complex macros to work. For these reasons, I am suggesting a new method called "of mixture", which overcomes Goodman's problematics and it is much simpler than all existing methods.

Practically, in presence of matrices of multiple columns or rows (or both), representing the votes for each party from an election to another, this new method doubles the relative compatible ones and replicates these in proportion of the votes of the electoral results. In order to do this, it is necessary to operate on rows or columns, transforming their values in base 100.

The new logical method "of mixture" obtains a standard error of 0.62%, calculated on the sum of the squares of the differences between the values of the estimated (TC) and effective (TT) row values, divided by the number of rows; this is higher than King and others' at 0.37%, but lower than Goodman's at 0.93%[29]. This result of 0.62% is acceptable both: 1) because its equivalent absolute error 8.06%[30], calculated for the 2018 elections, is lower than the acceptability threshold of 15% set by Corbetta Parisi and Schadee (Corbetta, 1988)[31], 2) in terms of trade-offs of the criteria exposed by De Sio (2008, p. 84-90) - extremely easy to calculate, replicability and having an acceptable and contained error -, 3) since the manipulation of the matrices has an accuracy of 99.4%, according to their mathematical properties (Abadir - Magnus, 2005).

Thanks to the proposed models it was possible to link the previous ideological positions of an election to another election through flow matrixes. With regards to the relation between electoral flows and the weighted ideological party distance, the model obtains an $R^2adj = 87.2\%$. This also allowed to reproduce the positional party competition of the years 1992 and 2018, creating a dataset, using as primary sources the ITANES archive (1948 - 2013) and others (Diamanti - Mannheimer, 1994, p. 114), (D'Alimonte R. - Chiaramonte A., 2010) (Bartolini - Chiaramonte - D'alimonte, 2002) (Carrieri, 2018) (De Sio - Paparo, 2014) (SWG, 2018) (IPSOS, 2018) (De Sio - Paparo, 2014).

1.3.4. Normative analysis of the main political variables

The third and last part of the thesis capitalizes on some previous links found in order to generate an optimization of the previous variables, and to use a game theory equilibrium between the strategies of party and elector interests.

In particular from 2013 in Italy the debate of how to determine the number of S has become more significant. In order to propose a solution, I hypothesize that the more demand for political representation without matching the political offer (by the parties presented at the election), the

[29] Calculation resulting from my analysis on an application of the model reported by De Sio (2009.1, p. 25).
[30] The value is calculated as 0.62% (standard error per line) * 13 (number of rows)
[31] They introduce the Vr indicator, which counts all the "impossible" coefficients because of the negative ones present in Goodman's flow matrix, placing the comprehensive tolerance threshold at 15%.

more illegitimacy of parties represented in the assembly, thus analyzing and also proposing tools of electoral engineering.

Chapter eight calculates the optimal value of S more accurately than the simple relation P=S^3 established by Taagepera (2007b, p. 199), and unprecedently it also formalizes the optimal value of M. Moreover, I introduce the original index D_E, which represents the dis-representation attributable to an electoral law and to the institutional shapes M and S (independent from N).

In order to minimize conflict channels – reported in the four pillars -, I propose two different methodologies, depending on the optimal variable I want to obtain. In the first methodology, I obtain the optimal values for S and M using differential calculus that minimizes the conflict channels produced by P, S and M, but also taking into consideration N merged with S, net of some parameters - as never done before -. In the second methodology, I obtain the optimal values for D_2 and D_E using maximization calculus between the cabinet duration and a new comprehensive representation index RE, which is equal to the complement of Gallagher's index of dis-representation $D_2 = [0.5\sum_{i=1}^{N_0}⟦(s_i - v_i)^2]⟧^{0.5}$ (1991), such that $RE = 1 - D_2$, which includes D_E and is standardized for the cabinet life (C) maximum duration.

The only genuinely exogenous variable is P, whereas the other variables (M, S, D_2 and D_E) are endogenous; N is to be considered a hybrid variable. All conflict channel formulas are derived for endogenous variables; the formulas are not derived for P or N, given their exogenous or hybrid nature. However, theoretically, it is possible to find the optimal value of N at the end of all optimization passages.

As for the first methodology, in supporting the introduction of N into the grafting of the assembly size (S) optimization, I considering that parties have the function of grouping people with a relatively common political view, and represent the base of cleavage formation (Heath, 2005) (Lipset and Rokkan, 1967) (Rae - Taylor, 1970) and of politics issues (Taagepera and Grofman, 1985), reducing de facto these said issues, and consequentially the conflict channels, versus simply considering S.

This reasoning is also applicable outside the chamber of representatives: vote behavior can also affected by common opinions among representatives on a program or ideology towards ideals and identity (Budge - Robertson - Hearl, 1987) (De Sio, 2011, p. 57-8), hence the parties. This implies that not all MPs effectively represent an active part in the determination of a conflict channel, therefore the Effective S will be defined between S and N, hence requiring the application of the weighted geometric mean of these.

Operatively, this means setting the following expression: $N^{(e)} * S^{(1-e)}$, with exponents *e* in [0, 1] and *1-e* as weights, calculated in function of their respective average values N and S. In detail, the parameter *e* is a theoretical parameter calculable by substituting the empirical values P, S, N, in the formula of S^*, for as many countries as possible, considering the most recent values available, and finally applying the geometric mean to the numeric values of *e* obtained for all countries. Following the same methodology, it is possible to obtain M^* by substitution of S^*.

Moving to the second methodology of maximization of the cabinet duration and R, I start by noting the relation between D_2 and the institutional variables M, S, and N such that $D_2 = \frac{0.5}{N^2} = \frac{0.5}{\sqrt[3]{MS}}$ (Shugart - Taagepera, 2017, p. 145-6). I can then substitute ${N_s}^2$ in the function $C = \frac{k}{{N_s}^2}$ with the geometric mean between: 1) ${N_s}^2$; 2) $\sqrt[3]{M^*S^*}$; 3) $1/2D_E$ (by definition). In such a way, I have nested M^* and S^* inside the previous calculations. I proceed to calculate the RE index, simply applying the basics of the probability for independent events, standardized for C. Finally, I obtain ${D_2}^*$ probabilistically from ${D_E}^*$.

Chapter nine casts into system the optimal values of $D_2, D_{El}, S\ and\ M$, using some others already available (R. Taagepera 2007b) (Shugart - Taagepera 2017) and some other formulas that I have founded here and the strategic vote, to determine n in function of these optimal values, and consequentially also the party and elector equilibrium's strategies. This is useful because: 1) from D_{El}, the impact can be evaluated of each political and institutional variable, like: M, S, legal thresholds of representation, majority premiums, proportional rules of allotment (quotient and divisors), and the other complicate electoral rules, also capitalizing the current literature (Taagepera R. , 2007) (Shugart - Taagepera, 2017), and then making electoral engineering; 2) it is possible to simulate and project electoral systems, evaluation of the seats' allotment in a particular country in case of changing of the electoral rules (threshold, majority premiums, proportional rules, etc.…), in function of the optimal values or not; 3) to find the optimal average of the ideological distance between parties, then knowing not only how the party shares' its configuration but also how the parties' ideologies are displayed on average.

Chapter nine wants to determine an equilibrium between parties' and voters' "electoral utility", which is the quantity of dis-representation which benefits a group of parties and voters in the system, producing disutility for the others. This chapter enriches the law of minority attrition including thresholds and majority premiums (MJPs) and strategic vote, using a primary game theory approach and the "Maximin" Rawlsian theory (1971) as a benchmark for equality.

This chapter provides: 1) a set of tools to determine party and elector equilibrium strategies; 2) a simulation and design of electoral systems, evaluating how seats' allotment varies with a change in electoral rules and then strategic votes in relation to threshold, majority premiums, but also any other corrective and proportional rule characteristic; 3) electoral projections through survey data, creating forecasts to understand, for example, if the mix of a specific electoral system with a specific political system goes to generate a majority in the assembly on not.

Chapter ten provides an overview of links among the new tools and knowledge developed in this thesis, with the final aim of the normative building of an optimal electoral system, which can warrant both logical coherence and social equity as categorized by Arrow (1951). I use the methodology "connection among connection", able to minimize the error (Taagepera R. , 2015, p. 89-95; 215-219).

For each given political system, optimizations can be performed by embracing the so-called "gradualistic engineering" approach, as a trial-by-error process that must follow the democracy, as it happens in a pure science (Antiseri, 2007, p. 520-1). The electoral system should be adaptive, therefore politics follow a trial-by-error process trying to find their stabilization, as a one-shot solution does not exist that is always valid, also according to Hayek's principle of exploration of the unknown and error correction (1982). The parallelism is similar to Dahl's reasoning - given by the homonymous "Dahl's box" - regarding the transition of political regimes, which must be gradual for a durable democratic transaction (1971).

All the tools introduced in this thesis can form a broad and interdisciplinary platform of contents and methodologies that can expand the knowledge in political science and generally in social sciences. For example, a vital but underestimated powerful tool in political and social sciences is given by the application of LQMs to political economy estimates (Taagepera R. , 2015, p. 224-8) and in particular by time series, even though, for instance, Bernhard and Leblang (2006) have offered a hybrid approach between politics, finance and the use of time series. Lastly, unfortunately in political sciences the differential calculus (Taagepera R. , 2015, p. 167-79) used, in this dissertation, in a comprehensive multivariate model (chapter eight) is less diffused, even if it is widely diffused in physics and economics.

1.4. Organization of the thesis

The thesis is organized as follows.

This first chapter has introduced the methodology of logical quantitative models and its applications to political sciences.

The second chapter explains the conversion of votes to seats. I use the law of minority attrition, expanding its form into a final model which is applicable from single member district to any electoral system in the analyzed period, yielding more accurate results also in the pure FPTP. I test it empirically for an in-depth analysis of Italian elections from 1992 to 2018, because of the peculiar complexity of this country over time. The resulting interlocked model uses both political and institutional variables, such as: the assembly size, population size, district magnitude, effective number of parties, threshold of representation. The model tested for Italian elections (354 cases) achieved an R squared adjusted equal to 97.5% between the actual and the estimated seats from votes, which makes it applicable to almost any electoral system.

The third chapter introduces the estimation of party seats from the previous elections using a weighted regression with independent variables jointly: 1) the product of the assembly size and the district magnitude, 2) the past values of the biggest party shares, and 3) the number of Effective parties and simply considered.
The second part of the thesis looks to widen the connections among the disproportionality index, positional party competition, electoral flows, and cabinet duration.

Chapter four develops a probability density function with five inflection points which describes any party system. It better catches the asymmetries among the party seats distribution at nationwide level. This function is developed starting from the law of minority attrition already introduced for the correlation between seats and votes, but in this case the exponent is expressed as a function itself, determining a change in its intended use. A concrete application can be the evaluation of the block threshold impact on the (seats) party system. A potential application of this function could be for a more accurate cabinet prediction as well as for the estimation of the seats allotted to each party in each district.

Chapter five implements the Downsian (or positional) competition model that describes the left-right space occupied by each party through Beta functions that I have tested on the Italian elections from 1992 to 2018.

Chapter six presents an in-depth qualitative analysis of the hypothesis that the more proportional an electoral system, the more the parties tend to centripetal competition, thus connecting ideological terms, effective number of parties and electoral system. The methodology used is the *comparative statics* (Hicks, 1939); (Samuelson, 1947), which here considers all equilibriums among the abovementioned variables, not considered as exogenous or endogenous, but as a multivariate interlocked system of causality. This allows to: 1) correlate how a specific electoral system can modify the party ideological positioning on the left-right continuum, 2) know how many parties would be in the political space, and their location, with no disproportionality due to electoral systems, 3) better analyze the countervailing effect of the electoral system in charge, introduced only qualitatively by Sartori (2003, p. 61-2). The correlation of the previous interlocked relation connects the positional distance and the effective number of parties (weighted on the electoral system).

In chapter seven, I suggest an alternative logical method to aggregate electoral flows, which resolves Goodman's problematics and provides a simpler solution than that of G. King. The empirical results obtained are acceptable, based on the criteria established by De Sio, Corbetta Parisi and Schadee. Capitalizing on this method, I proposed an extensive application allowing to link the previous ideological positions of an election to another election through flow matrixes. With regards to the relation between electoral flows and the weighted ideological party distance, the model obtains an $R^2 adj = 87.2\%$. This also allowed to reproduce the positional party competition of the years 1992 and 2018.
In the last part, the third one, I capitalize on some previous links found, optimizing the previous variables using physics, mathematical and game theory tools.

Chapter eight provides tools to more accurately calculate the optimal value of S (Taagepera and Shugart, 1989, p. 175), and unprecedently, the optimal value of other institutional variables such as: the district magnitude, the Gallagher's index of dis-representation, and the dis-representation index attributable to an electoral system (D_E), originally developed in this thesis. An innovative finding presented in this chapter is that the higher N_s and/or M*S* product, the more ${D_E}^*$ tends to 0; conversely, the lower N_s and/or M*S* product – both tending to 1 -, the more $D_E{}^*$ tends to 1, describing the correlations between N_s and $D_E{}^*$ and between M*S* and $D_E{}^*$ as branches of the hyperbola.

Chapter nine wants to determine an equilibrium between parties' and voters' "electoral utility", which is the quantity of dis-representation which benefits a group of parties and voters in the system, producing disutility for the others. This chapter enriches the law of minority attrition including thresholds and majority premiums (MJPs) and strategic vote, using a primary game theory approach and the "Maximin" Rawlsian theory (1971) as a benchmark for equality. This chapter provides: 1) a set of tools to determine party and elector equilibrium strategies; 2) a simulation and design of electoral systems, evaluating how seats' allotment varies with a change in electoral rules and then strategic votes in relation to threshold, majority premiums, but also any other corrective and proportional rule characteristic; 3) electoral projections through survey data, creating forecasts to understand, for example, if the mix of a specific electoral system with a specific political system goes to generate a majority in the assembly on not.

Chapter ten provides an overview of links among the new tools and knowledge developed in this thesis, with the final aim of the normative building of an optimal electoral system, which can warrant both logical coherence and social equity as categorized by Arrow (1951). I use the methodology "connection among connection", able to minimize the error (Taagepera R. , 2015, p. 89-95; 215-219). For each given political system, optimizations can be performed by embracing the so-called "gradualistic engineering" approach, as a trial-by-error process that must follow the democracy, as it happens in a pure science (Antiseri, 2007, p. 520-1). The electoral system should be adaptive, therefore politics follow a trial-by-error process trying to find their stabilization, as a one-shot solution does not exist that is always valid, also according to Hayek's principle of exploration of the unknown and error correction (1982). All the tools introduced in this thesis can form a broad and interdisciplinary platform of contents and methodologies that can expand the knowledge in political science and generally in social sciences. Lastly, unfortunately in political sciences the differential calculus (Id. (p.167-79)) used, in this dissertation, in a comprehensive multivariate model (chapter eight) is less diffused, even if it is widely diffused in physics and economics.

PART I SEATS AND VOTES AND THE LOGICAL QUANTITATIVE APPROACH

Chapter 2

Implementing the seats from votes relation: the Italian case 1992-2018

2.1 Introduction

Notwithstanding intuition could logically induce to think that votes obtained in an election are a good indicator of the seats representing the final political representation, this is not always correct. As introduced before, nowadays, even in excellent and mature democracies, minority parties or coalitions (in terms of votes) could be able to win elections: the more majoritarian the electoral system, the more the wider alliance of parties is strategic to determine the success of those. How is this possible? The answer is that aside a pure proportional system which tends to a non-strategic competition (even though also proportional formulas are never impartial in the seat allotment), on the other hand, majoritarian electoral systems go to improve some parties' particular interest against others, and generally it goes to increase governability, counting and grouping votes in some particular way, in order to increase their final seats in the assembly. These could be the results of intended or unintended consequences of the electoral reforms approved by parties in parliament. It is like the sentence "private vices, public benefits", the subtitle of Mandeville's "The Fable of the Bees" (1714), but in this case, I propose the respect of the following logical asserts: the more the N, the less the cabinet stability; a wider seats-votes gap produced by the electoral system in favor of the relative majority party/coalition (in term of seats) increases the cabinet stability under the condition that the sum of the gaps among all parties is the least possible. Thus, this chapter finds a unified model able to synthesize electoral systems through linking votes and seats, and the next chapter will go to solve that axiom quantitatively.

Hence, here I show the implementation of the sigmoid law of minority attrition, which interlocks the political and institutional variables; this will allow us to apply the final model to any electoral system and functioning even better in the pure FPTP as used until now by Taagepera (2007b, p. 232). Using this "comparative statics", the final equilibrium is then obtained thanks to a multivariate interlocked causality system rather than using exogenous variables. It will analyze and project all possible scenarios, such as making more efficient exit polls, electoral engineering and political strategies. Actual knowledge about these tools among the political scientists - besides Taagepera - is stopped to the simple directional relations, and not quantitative ones (Fisichella, 2009, p. 263-288) (Grofman, 2004) (Sartori, 2003; 1987) (Colomer J. , 2004; 2005).

The aim is to develop a general model of conversion, obtained through an in-depth analysis of the Italian elections. I have chosen the Italian context because of the peculiar complexity of this country over time, presenting: thresholds, majority premiums, the simultaneous proportional and the district's single winner (FPTP) and other fuzzy compensative mechanisms of seat allotment. All these peculiarities allow to build a generalized model that is applicable at a cross country level (excluding alternative vote, single transferable/non-transferable vote, and their derivates)[32].

The law of minority attrition works well for simple scenarios represented by FPTP, and when there is a bipartite system like in the Caribbean Islands, but less if: 1) N is high; 2) M and-or S are low; 3) the electoral laws are complex, when for example the electoral system is mixed, like in Italy, Spain, Japan and others. Then the aim is to system:

1) N in substitution of the number of the opponents competing for the specific election (one or two (Taagepera R. , p. 207-9));

2) other political and institutional variables interlocked to the law of minority attrition, modifying the degree of dis-proportionality - the exponent n - between seats and votes, allowing to apply the final model to any electoral system, and functioning even better in the pure FPTP.

I start from Taagepera's generalization of the law of minority attrition, which foresees only 1 and 2 opponents in the election; to implement it, I introduce N_v at the denominator in substitution of the numeric constant representing the opponents. The other important part of the original law of minority attrition formula is the parameter n, defined in the interval [1, ∞]; to guarantee such more comprehensive application, I propose to adopt implementation of the index n, which will become n_1 by means of the other institutional and political variables P, S, M, N, E, and a new index G (soon explained). The n_1 will be adjusted, obtaining a new and final n_1, following the same logical-quantitative methodology base. Being n=3 the ideal-typical value that n must assume for the FPTP, in the presence of some specificities of the electoral law, n_1 could be furtherly

[32] PBV, AV, STV, SNTV, MNTV, MMM, MMP, see Shugart and Taagepera for this classification (2017, 31-60)

and integrally transformed in the final n_2, especially when n can logically (potentially) degenerate in 1 or ∞. These cases happen in the presence of: 1) block threshold impact - T - (degenerating in ∞); 2) correctives used in the mixed electoral system, such as the proportionality effect of the "*scorporo*" s (soon explained) or a compensative mechanism of proportionality (both degenerating in 1), to the majority premium (degenerating in ∞).

Unprecedently in literature, I propose the new concentration index G, indicating the Effective Gini Index, aiming to measure the parties' vote concentrations within a country. It quantifies the party system's degree of structuring based on each party's concentration within the territory on a district basis. G is a logical mix of three indicators: N_0, N_2, and the classical Gini index g. The aims are to neutralize the impact of small parties over the effective ones $(N_0 - N_2)$ conditional to N_2 and depurated of the impact of N_0. G_d's final result consists of calculating the latter three variables referring to each party's votes distribution within the districts; finally, the weighted average of the G for the parties' share will be calculated, indicating the parties' average territorial concentration.

Hence all the previous variables will be mixed on the following logical-deductive assumptions:

1) Considering the discrepancies existing in the electoral systems from the equality relation P=S^3 (seen before), the higher P respect S^3, the higher n, and vice versa.

2) The more N_2, the less the other parties' marginal power to win a seat (among the districts), the lower n because of the parties' coordination problems.

3) The higher G_d, the lower n, because other parties' probability to win a seat (among the districts) is higher due to the specific territorial golden shares of party (or parties).

4) A corrective based on M conditional to S. The more M tends to S, the more the electoral system tends to be proportional – getting n lower – on the opposite, for M=1 the electoral system tends to be a pure plurality, thus getting n higher.

5) A corrective based on E conditional to P. The more E tends to P, the less is n, because the territorial distribution can less impact

the dis-proportionality. Conversely, the more E tends to 1, the higher n.

6) Considering the national block threshold of representation T or majority premium, these can nullify any proportionality of the electoral system, thus intervening at the end on the whole n_1, transforming – increasing it – in the final n_2. The more T tends to 1, and the more n_2 tends to ∞. On the opposite, the less is T, and the more n_2 tends to n_1. The *scorporo* follows the opposite dynamic.

It is useful to introduce the complex proportional mechanism, the *scorporo* – a corrective variable used in Italy from 1994 to 2001 – which consists of subtracting votes from the winning parties in the FPTP seats' allotment (equal to ¾ of S) to benefit the other lists for the proportional section (equal to ¼ of S).

Hence, concerning the conversion of votes to seats for the in-depth analysis of the Italian case, the elections that happened from 1992 to 2018 are taken into consideration in a postdictive analysis because these configure four different mixed electoral systems[33] representing a considerable dynamism of the party system both genetically, numerically and in terms of mechanic competition, useful to codify these complex schemes and reapply to generalize these to other countries. Then I used 354 cases obtained by data from the Interior's ministry.

The model tested for Italian elections has achieved an R^2adj=97.5% between the actual and estimated seats from votes. It has produced an error inferior to 0.601 percentage points compared to the simple law of minority attrition, and less than 27.8 % in absolute terms.

2.2 Logical Models and their implementations

Another concentration index that will be used here, with different characteristics from HH, is one derived by Gini (g). This index maintains the domain [0,1], where 0 indicates the fair distribution of all party shares in the system, and 1 indicates the maximum concentration, where one party holds 100%. Being g a concentration index, it must presuppose a N_0 not higher than 1 as it is given by definition, but at least equal to 2

[33] *c*overing the PR, MMM, and MMP types (having the FPTP, the uses of T, s and the MJP).

to measure the inequality in a meaningful way, otherwise if N_0 was equal to 1, the g value would degenerate in 1. Moreover, to calculate g, it is required to sort the quotas p_i such that $p_i < p_{i+1}$, and with $f_i = \frac{\mathrm{i}}{\mathrm{N}_0}$ indicating the maximum quota for positional lag, I get:

$$g = \frac{2 * \sum_{i=1}^{\mathrm{N}_0} \left(f_i - \sum_{\mathrm{j}=1}^{i} p_j \right)}{(\mathrm{N}_0 - 1)}$$

The g index only considers the equal party shares' distribution among the actors in a system, while the HH index considers the magnitude of the ideal-typical party share or "median" party, representing the entire system. Exemplifying: if we had five parties in a system, and each of them obtained 20% of votes, the g index would be equal to 0, so it would not capture the value of the share held by each actor; whereas, the HH index in the same scenario would be equal to 0.2.

2.2.1 For a New Index of Concentration and Territorial Distribution: The Effective Gini Index

Due to the different properties of the g and HH indexes, the need to build a new index arises that can system these and overcome the limits of g, when considered on its own. These limits are:

1. sensitivity to *outliers* $(\mathrm{N}_0 - \mathrm{N}_2)$;
2. impact of N_2 not capitalized ceteris paribus for the $(\mathrm{N}_0 - \mathrm{N}_2)$;
3. sensitivity to N_0.

With regards to the first limit, let me consider the following example with three quotas: 25%, 35% and 40%, which have a Gini index of 0.15. If another party is introduced, holding a quota of 1.2%, and rearranging the distribution, for instance getting the following set: 38.8%, 35%, 25%, 1.2%, the Gini index will be equal to 0.409. Therefore, N_0 is the main responsible for this substantial variation that I want to mitigate, considering the gap $\mathrm{N}_0 - \mathrm{N}_2$ (inserted at the denominator); to warrant the G_2's domain in [0,1], and avoiding the division by 0, in addition I introduce at the denominator the element +1 since theoretically $\mathrm{N}_0 - \mathrm{N}_2$

could be equal to 0. Now I set G_2 as follows:

$$G_2 = \frac{g}{\mathrm{N}_0 - \mathrm{N}_2 + 1}$$

G_2 represents the "Efficacious Gini Index", which indicates the effective concentration in a system net of the non-effective number of parties (N_0-N_2). Using the information in the previous example, G_2 is equal to 0.203, which is higher than 0.15, due to the introduction of the additional party, and lower than 0.409, due to the additional party holding a lower quota than the others. In the figure 6 below, I have reported the behavior of G_2 conditional to the non-actual number of parties.

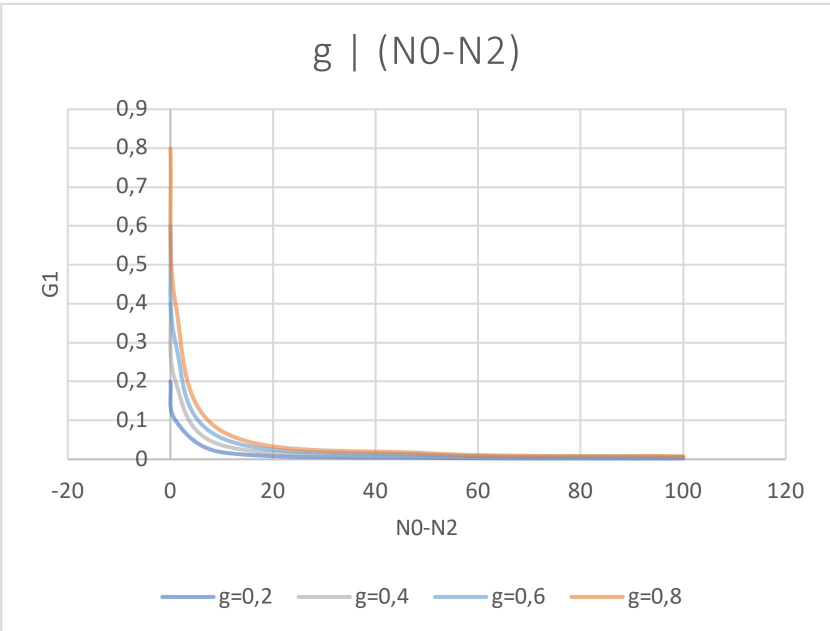

Figure 6 The new "Efficacious" G_2, calculated from the Gini index g, net of the non-effective number of parties ($N_0 - N_2$).

The "Efficacious Gini Index" defined above overcomes the first limit of the g index listed above. With regards to the second limit, I propose another step aimed at making the impact of the non-effective quotas less drastic, placing N_2 as benchmark. The previous considerations regarding the division by 0 and the domain apply, adding the element +1, because the object $\frac{\mathrm{N}_0-\mathrm{N}_2}{\mathrm{N}_2}$ for $\mathrm{N}_0 - \mathrm{N}_2$ equal to 0, would be equal to 0 too; therefore the final result is as follows:

$$G_1 = \frac{g}{\left[\frac{\mathrm{N}_0 - \mathrm{N}_2}{\mathrm{N}_2}\right] + 1}$$

In the figure 7 below, I have reported the behavior of G_1 calculated from the Gini index g, net of the non-effective number of parties $(\mathrm{N}_0 - \mathrm{N}_2)$ and the non-effective quotas. The curves represented in this graph are smoother and flatter than in the previous graph for equal values of g.

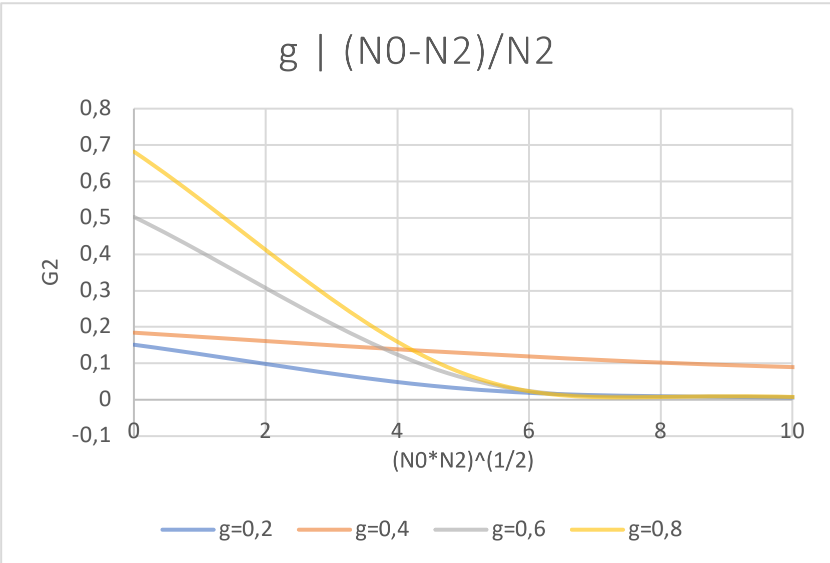

Figure 7 The G_1, calculated from the Gini index g, net of the non-effective number of parties ($N_0 - N_2$) and the non-effective quotas.

With regards to the last limit of g, I am looking to avoid that an actual index Gini "G" can be invalidated according to the simple application's context. To do so, the number of nominal dimensions N_0 that make the context up need to be considered jointly with N_2, to produce the most stable and attenuated formulation of Gini. The previous considerations regarding the division by 0 and the domain apply, adding the element +1, because the object $\left[\frac{N_0-N_2}{N_2}\right]^{\left(\frac{N_2}{N_0}\right)}$ for $N_0 - N_2$ equal to 0, would be equal to 0 too; therefore the final result is as follows:

$$G = \frac{g}{\left[\frac{N_0 - N_2}{N_2}\right]^{\left(\frac{N_2}{N_0}\right)} + 1}$$

The figure 8 below shows that the new G Index is fully independent from N_0 and N_2.

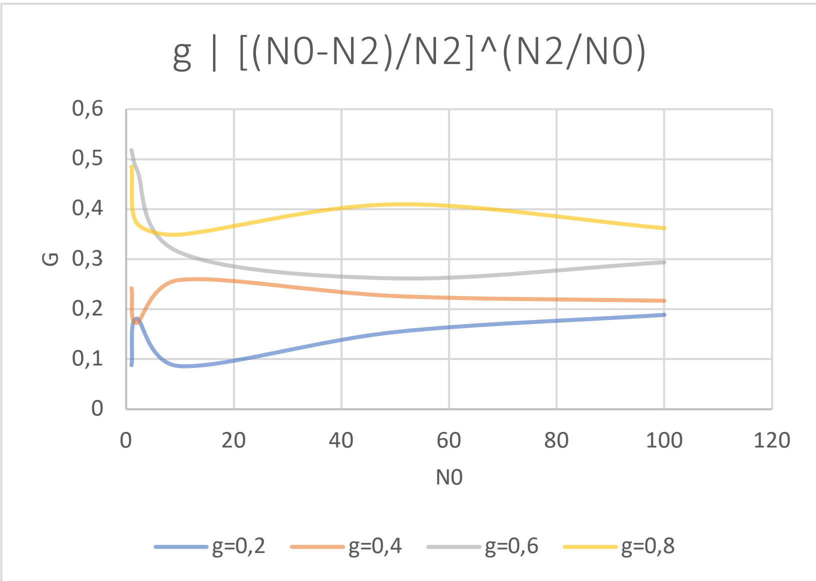

Figure 8 The Effective Gini index G, calculated from the Gini index g, net of the non-effective number of parties ($N_0 - N_2$), the context (N_0), and the non-effective quotas.

The G-index will find application to assess the impact of the electoral threshold and measure how much each party turns out to be concentrated on the national territory G_d. The minimum unit on which to calculate the variables g, N_0, and N_2, is the regional percentage obtained from each party. To obtain the G_d overall result, a $G_{d,p}$ is calculated for each party p_i, and subsequently their weighted average is applied. This allows to catch the effective local concentration of the parties net of symbolic candidatures, in which the party's impact is marginal in the system.

2.3. Converting Votes in Seats: from the Law of minority attrition to a new model

I now turn to the heart of the problem, which is how to calculate a mathematical function that can be valid to convert the votes to seats in any electoral system. I use the logical-quantitative model proposed by Taagepera as a starting point and I propose additional grafts, adjustments and implementation to predict the various specificities of a mixed or straightforward electoral system, thus generalizing it as much as possible to allow application in other contexts.

I restart from the last logical generalization of the law of minority attrition:

$$s = \frac{v^n}{v^n + (N_v - 1)^{1-n}(1-v)^n}$$

Although this model is an excellent starting point, it does not take into consideration other variants of voting systems such as the barrage thresholds or the territorial concentration of parties, as said in the first paragraph[34]. As I will show in more details later on, even though Taagepera (et al.) formalized these variants, they are not blended in the formula above. Moreover, the Italian case allows me to introduce additional elements, for example: the effect of the "scorporo" (which I will define later), plurinominal districts[35], and the peculiarity of mixed electoral systems, which have two sides of seats allotment. Specifying

[34] (Sartori, 1987, p. 58-61) (Rae, 1971, p. 95) (Riker, 1982, p. 760)
[35] which however generate dynamic plurality if exceedingly small.

this last point, the majority of electoral laws in force in Italy in the reference period, is characterized by a proportion of the seats' quota allotted proportionally to votes and another by plurality.
Starting to modify the law of minority attrition, I consider the "optimum" value that n must take in a pure plurality, which is 3, and rearrange the law as follows:

$$n = \left[1 + \frac{2}{0{,}5\left(1 - \frac{P - S^3}{P * S^3}\right)} * \frac{(1 - G_d)}{(\mathrm{N}_2 - 1)}\right]^{\frac{S-M}{SM-1} * \left[1 - \frac{E}{\sqrt{P}}\right]}$$

This can happen on the basis of the following logical-deductive assumptions:

1) where a sub-representation exists such that $P > S^3$, then the dis-representation index n will increase; inversely, for $P < S^3$ I expect a decrease of n down to 1 (for $S \rightarrow \infty, and\ P \ll S$). In fact, the object $\frac{2}{0{,}5\left(1-\frac{P-S^3}{P*S^3}\right)}$ is defined in $[0, \infty]$ because by definition $P \geq$ S.

2) the higher the average territorial concentration of the parties on territory G_d, the lower n (for $G_d = 1 \Rightarrow n = 1$). For example, in a system of two districts, both with 100 voters, party A gets 70 votes in the first district and 30 in the second, party B gets 30 votes in the first and 40 in the second, and party C gets only 30 votes in the second; ceteris paribus, party A will win only one seat, being more concentrated than if it had had 60 votes in the first district and 40 in the second, where it would win both districts.

3) Capitalize N_2. Considering the limit case in which N_2 tended to the anchor point 1 - but never reached it, because $\mathrm{N}_2 = 1$ configures a dictatorship - even the remaining party shares on the territory, although concentrated, would see a reduced probability of being able to win even one district because the presence of the bigger party would vandalize it, then $n \rightarrow \infty$. On the other side, if N_2 tended to infinity, the competitive numeric

advantage for every party would be null even in the case of uninominal districts. The nullification happens because the winning probability in each district is almost identical for each party, as the victory will be determined for someone randomly thanks to a handful of votes for each district, then $n \to 1$.

4) The first member of the exponent, $\frac{S-M}{SM-1}$, aims at assessing the dimension of districts M, thus allowing the application of the model to several values of M coexisting in the same electoral system, and their relationship with S. Through a logical study of the limit cases, it is possible to deduce that for S = M there is a single nationwide district, in which the dis-proportional effect given by the mechanism of the districts would be null. On the opposite extreme, for M = 1 in presence of a pure plurality, the dynamics of the districts would fully exert their effect.

5) The second member of the exponent, $1-\frac{E}{\sqrt{P}}$, takes the population size into consideration for the first time. I have used E instead of S considering the relationship $E=\frac{S}{M}$, which indicates the population's share represented by an MP. Knowing that $1 < E < P$, and knowing that a high E implies small districts, the higher E, the lower n, then the smaller the districts, the more n tends to 1. Hence, I have defined the square root of P, applying the geometric mean of the previous domain [1, P], such that $\lim_{E \to P}\left(1-\frac{E}{\sqrt{P}}\right)=0$.

2.4. A comprehensive model: the Italian case 1992-2018

Having enunciated the general model theoretically, I can now apply it to concrete cases, in particular Italy from 1992 to 2018 – with further correctives and integrations – to test it empirically.

When an electoral system foresees a joint proportional and majoritarian allotment, it is necessary to articulate the variable N_2 into N_p and N_{cl}, both calculated on the basis of votes. The first simply reflects the effective number of parties, while the latter reflects the coalitions formed by the same parties. These two variables will be applied to each allotment of the electoral system respectively (N_p for the proportional and N_{cl} for the majoritarian representation).

Similarly, the M dimension is articulated into M_c, referring to constituencies and M_d, referring to districts. This is a helpful distinction in the joint use of both variables in mixed electoral systems.
In the 1992 election, the Italian electoral system was proportional for both the chamber and the senate. Therefore, N_2 needs to be articulated into N_p and N_{cl}, and M into M_c and M_d.

In 1994, 1996, and 2001, the electoral system was plurality for about 3/4 (74.52%) and proportional for 1/4 (25.48%).

The elections from 2006 to 2013 used a mixed electoral system with a proportional base or repartition, majority premium and block thresholds. The country's constituencies allotted 618 seats proportionally, whereas 12 were attributed to four foreign constituencies.

In 2018 the electoral system was plurality for 37%, proportional for 61% and the remaining 2% was allotted to the Italian voters abroad, distributed into four foreign constituencies. The electoral system in 2018 was also characterized by M equal to 3 (seats) for the chamber and 1.5 for the senate. Therefore, being $N_p < M$, I can consider N_{cl} for the foreign seats' allotment, hence the party competition is majoritarian.

For simplicity, it is possible to consider foreign competition assimilable to the district dimension, being the M dimension much nearer to the district rather than the constituency one. Hence, the foreign seats' allotment implies a simple substitution of M with M_d.

I now rewrite the new intermediate indexes n_1 for each election.

$$n_{1,1992} = \left[1 + \frac{2}{0,5\left(1 - \frac{P - S^3}{P * S^3}\right)} * \frac{(1 - G_d)}{(N_p - 1)}\right]^{\frac{S - M_c}{SM_c - 1} * \left[1 - \frac{E}{\sqrt{P}}\right]}$$

$$n_{1,1994-2001} = (1 - 0.2548) * \left[1 + \frac{2}{0,5\left(1 - \frac{P - S^3}{P * S^3}\right)} * \frac{(1 - G_d)}{(N_{cl} - 1)}\right]^{\frac{S - M_d}{SM_d - 1} * \left[1 - \frac{E}{\sqrt{P}}\right]} + 0.2548\left[1 + \frac{2}{0,5\left(1 - \frac{P - S^3}{P * S^3}\right)} * \frac{(1 - G_d)}{(N_p - 1)}\right]^{\frac{S - M_c}{SM_c - 1} * \left[1 - \frac{E}{\sqrt{P}}\right]}$$

$$n_{1,2006-2013} = 0.9810\left[1 + \frac{2}{0,5\left(1 - \frac{P - S^3}{P * S^3}\right)} * \frac{(1 - G_d)}{(N_p - 1)}\right]^{\frac{S - M_{c(n)}}{SM_{c(n)} - 1} * \left[1 - \frac{E_{(n)}}{\sqrt{P}}\right]} + 0.1190\left[1 + \frac{2}{0,5\left(1 - \frac{P - S^3}{P * S^3}\right)} * \frac{(1 - G_d)}{(N_{cl} - 1)}\right]^{\frac{S - M_{c(e)}}{SM_{c(e)} - 1} * \left[1 - \frac{E_{(e)}}{\sqrt{P}}\right]}$$

$$n_{1,2018} = 0.39\left[1 + \frac{2}{0.5\left(1 - \frac{P - S^3}{P * S^3}\right)} * \frac{(1 - G_d)}{(N_{cl} - 1)}\right]^{\frac{S - M_d}{SM_d - 1} * \left[1 - \frac{E}{\sqrt{P}}\right]} + 0.61\left[1 + \frac{2}{0.5\left(1 - \frac{P - S^3}{P * S^3}\right)} * \frac{(1 - G_d)}{(N_p - 1)}\right]^{\frac{S - M_c}{SM_c - 1} * \left[1 - \frac{E}{\sqrt{P}}\right]}$$

For the election of the chamber and the senate of 1992 we can simply consider the final n_2 equal to n_1, because neither block thresholds, majority premium nor relevant district allocations[36] occur. Therefore:

$$n_{2,1992} = n_{1,1992}$$

For the elections from 1994 to 2001, I must also consider the peculiar expediency of the "*scorporo*"[37] that could be translated as "unbundling", which impacted N significantly, compared to other more marginal elements of that electoral law such as the *repêchage*[38]. The *scorporo* at the chamber of deputies was "partial": this consisted of allocating seats in the proportional share subtracting from each party the votes obtained by the first unelected (or runners-up) in the districts (within a given constituency) in which the former had won. In the senate, the *votes' scorporo* was "total": all votes obtained by an elect in the district were

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[36] The unified electoral law of March 30th, 1957, n°361, established that where a party had overtaken the 65% of votes of a relative district, it would get directly one of the 315 seats provided for the senate. However, this hypothesis has been concretely very marginal in the history of the republic; in 1992, just two seats were attributed in this way, therefore producing a disproportionality index n equal to $n = [1/(1 - 2 * 0{,}35/315)] = 1{,}002$ at most, which is absolutely marginal, and that I do not consider for simplicity. However, if another electoral system diminished the threshold of winning the district ballot, or the G_d had been much higher, the probability of winning the district would be higher for small parties.

[37] "Mattarella Law" of August 4th, 1993, N° 276 (N° 277 for the chamber of deputies).

[38] Which established the direct election in a proportional quote of a limited representatives' number (3 or 4, depending on the circumscription's extent); the rest is elected by the repêchage in the uninominal districts - among the "best losers" candidates – by implementing the value of n, but in a residual way.

subtracted from the same list in the proportional dimension (of the constituency).

All these will result in an s index representing the impact of the *scorporo* in the interval [0,1], which will be inversely proportional to n, as s plays a proportionality role. As the first and the second party are expressly envisaged by the electoral law, I propose probabilistic tools to estimate them.

I already know that $p_{1,\mathrm{T}} = \frac{1}{N^{\frac{3}{4}}}$, than $\mathrm{N} = \frac{1}{p_{1,\mathrm{T}}^{\frac{4}{3}}}$ (Taagepera R. , 2007b, p. 148-155). However, in this case, the exponent is fixed and found empirically without a genuine logical construction. This offers an opportunity to parametrize the exponent of N, in this case N_{cl}, as it is the coalitions that compete in the districts and not the individual parties (although there are parties that do not cooperate). We can hypothesize that the greater the actual concentration of quotas, here considered at the national level and no longer on a territorial-regional basis, the greater the first party p_1; p_1 exists for each value of G defined in [0,1], therefore the denominator is always different from 0. This can be formalized as follows:

$$p_1 = \frac{1}{N_{cl}^{1-G}}$$

At this point, I replace p_1 in the following formula, already found by Taagepera (2007b, p. 154) (readapted):

$$p_2 = \frac{\left(1 - p_1\right)}{\sqrt{N_{cl}^{\frac{3}{2}} - 1}}$$

At this point, I apply n_2 to the chamber and the senate to distinguish the two different "*scorpori*", "s_c" for the chamber and "s_s" for the senate, both having domain [0,1] (producing a proportionality effect):

1) for the chamber of deputy, I consider only the weight of the second party and those of the other parties (grouped by approximation, excluding the first and second);

2) for the senate I consider the first, the second and the remaining others (by approximation, as done above).

For both the chamber and the senate, the dimensions are squared because it has the meaning of a weighted average where the weights are equal to the same quotas. Both *scorpori* will be multiplied by 0.25 as this represents the electoral system's quota affected.

So, we will have that:

$$s_s = 0.25 * \left({p_1}^2 + {p_2}^2 + \left(1 - p_1 - p_2\right)^2 \right)$$
$$s_c = 0.25 * \left({p_2}^2 + \left(1 - p_1 - p_2\right)^2 \right)$$

I also need to consider the block threshold Ti, which excludes parties that receive a number of votes below a specific value T. This threshold has an impact on the final coefficient of disproportionality n_2: the bigger the threshold T, the higher n_2; vice versa for T equal to zero, n_2 will equal n_1 ceteris paribus. The block threshold Ti impacts n_2 proportionally to the euclidean distance[39] that elapses between the ideal-typical party share HH (equal to the reciprocal of N calculated by the coalition quotas) and the ideal-typical party share which considers the entire proportional party system, knowing by definition that ${N_{cl}}^{-1} > {N_p}^{-1}$, hence obtaining a block threshold impact equal to the object $\left(1 - \left({N_{cl}}^{-1} - {N_p}^{-1}\right)\right)^2$ in the domain [0,1].

The objective is to formalize the strategic element of the list's composition into the districts. My model captures the quantitative elements of the party system. However, the political system includes human aleatory variables, such as party alliances, which are challenging

[39] (Abadir - Magnus, 2005, p. 1-3)

to formalize in a quantitative model. For example, some small parties would have no chance of either conquering the district or reaching the threshold in proportional alone, but they can ally with the main coalition's party(ies), thus managing to get their candidates elected.

I finalize the formula to calculate n_2 for the chamber and the senate in relation to s, n_1 and Ti as follows:

$$n_{2,1994-2001} = \frac{n_{1,2018}(1-s)+s}{1-Ti\left(1-\left(N_{cl}{}^{-1}-N_p{}^{-1}\right)\right)^2}$$

The numerator has been built thinking at the limit case for which the maximum s is 1; this is possible in absolute terms, but not in this concrete case where it might only reach 0.25.

I can now formalise the impact of T, as a proxy of disproportionality n_2, with the Ti formula in domain [0,1], using the index G_p - the Effective Gini index applied to parties on a regional basis – applicable to all elections:

$$Ti = 1 - \frac{1}{TN_p{}^{1-G_p}\frac{1-G_d}{\sqrt{\frac{S}{M_c}}}+1}$$

With regards to the first component of the denominator, $TN_p{}^{1-G_p}$, when T equals 0, the impact on disproportionality will also be zero. On the opposite extreme, when T equals 1, there are two limit cases:

1) for N_p equal to 1 and G_p equal to 1, the impact would be equal to 0.5. In fact, this scenario could represent a party regime tending to a single-party, where an opponent could theoretically get at most 50% so not to put the biggest party at risk of losing the election. In other words, using a game theory reasoning, 0.5 is precisely the point where the pay-off to the larger party would maximize its electoral profit. This extreme scenario could be concretized by a situation where the biggest party also holds legislative powers.

2) for N_p tending to infinity and subsequently G_p logically tending to 0, the impact would be equal to 1. In fact, in this scenario, a hypothetical block threshold of 1 would have an impact that inevitably would involve all parties, therefore $Ti = 1$.

With regards to the second component, $\frac{1-G_d}{\sqrt{\frac{S}{M_c}}}$, this aims at putting to system the impact of the territorial "diffusion" $(1 - G_d)$, as complement to the concentration index G_d, which is inversely proportional to T as the more concentrated the parties' votes, the more ineffective the threshold, conditional to the number of districts $E = \frac{S}{M_c}$.

For the chamber of deputies, E will be equal to 1, since $M_c = S$, considering that the threshold is set on a national basis. On the contrary, as the senate is elected on a regional basis, the territorial parties' votes diffusion $(1 - G_d)$ diminishes, thus resulting in a reduction of the T effect; this happens because the probability for a party to reach the threshold in at least one region increases with E. Being this a random probability that ranges from 1 to E, then I can calculate the geometric mean between them, taking the denominator of the second component to $\sqrt{\frac{S}{M_c}}$.

For the elections of 2006, 2008 and 2013, I must introduce an additional corrective Mjp defined in [0,1] based on the Majority Premium. Following the first law[40] of Duverger (1951; 1954), I can assume a simplified scenario in which on average, a majoritarian competition tends to two coalitions that compete for the Premium.

In the case of the chamber of deputies, the Majority Premium is granted to the most voted list or coalition that does not reach 55% of votes at the national level, which gets 340 seats. The net gain for the first party-

[40] called in this way by Riker (1982).

coalition $\left(0{,}5397 - p_1\right)$[41] could be a suitable corrective Mjp_c for the chamber.

The same rule applies to the senate but on the regional level. The corrective Mjp_s needs to measure how much the probabilistic gain is for the party-coalition beneficiary, reflecting the average Majority Premium granted at regional level in just one formula. For this reason, I need to include the territorial distribution's impact of this competition, which I formalize with the exponent $\frac{G_d}{\sqrt{\frac{S}{M_c}}}$.

This exponent is applied to the first party-coalition's competitive advantage $\left(p_1 - p_2\right)$ to win the regional Majority Premia on average. In fact, for a perfect equal territorial distribution, the exponent tends to zero, implying that the average regional Premium is equal to the difference between each regional Premium (0.5397) and the biggest party-coalition's share $\left(p_1\right)$, since there is no variance among each regional vote distribution. I have used $(p_1 - p_2)$ to measure the exclusivity of Mjp in each region, which would otherwise be nullified in case the biggest party-coalition received a number of votes very close to the runner-up. In fact, in this case, the increase of the territorial party concentration would offer the runner-up the opportunity to win regional Premia from a tie, thus increasing the variance at the national level. When the variance of the regional vote scenarios increases, these scenarios will tend to the national vote distribution and the competition dynamic will align to the national one, thus nullifying the Mjp_s.

Therefore, the formulas for the corrective Mjp are as follows:

$$Mjp_s = \left(0{,}5397 - p_1\right) * \left(p_1 - p_2\right)^{\frac{G_d}{\sqrt{\frac{S}{M_c}}}}$$

$$Mjp_c = \left(0{,}5397 - p_1\right)$$

[41] The constant is 0.5397 because of the 2% of seats distributed among the foreign constituencies.

I now move to calculate $n_{2,2006-2013}$, applicable to both the chamber and the senate, in the same way as done for $n_{2,1994-2001}$with two differences: I exclude the *scorporo* which was not present in the electoral system of those years, and I add the conditional probability for independent events (Espa - Micciolo, 2008, p. 51), which considers the threshold impact and the Mjp jointly. The resulting formula is:

$$n_{2,2006-2013} =$$

$$\frac{n_{1,2006-2013}}{1-\left(Ti*\left(1-\left(N_{cl}{}^{-1}-N_p{}^{-1}\right)\right)^2+Mjp-\left(Ti*\left(1-\left(N_{cl}{}^{-1}-N_p{}^{-1}\right)\right)^2\right)*Mjp\right)}$$

For 2018, I exclude the corrective Mjp since Majority Premium was not present in the electoral system for that year:

$$n_{2,2018} = \frac{n_{1,2018}}{1-Ti\left(1-\left(N_{cl}{}^{-1}-N_p{}^{-1}\right)\right)^2}$$

2.5. The empirical test on the aggregated disproportional index $\mathbf{n_2}$

After this detailed construction of the model, we can finally test it by comparing the estimated seats from votes with the real ones. I have taken the chamber’s and senate's results of all parties from all elections in the period 1992 - 2018, obtaining a sample of 354 data points. All the variable that we need to substitute into the previous formulas are listed in the tables 3 and 4 below:

Table 3 Political and institutional specific variables for the Italian elections 1992-2018.

Elections	Assembly	Np	Ncl	G	Gd	N0	Mc	S
1992	Chamber	6.62	6.62	0.33	0.131	28	19.69	630
1992	Senate	7.04	7.04	0.34	0.131	33	15.75	315
1994	Chamber	5.42	2.81	0.30	0.171	19	23.75	630
1994	Senate	5.42	3.13	0.38	0.171	19	11.60	315
1996	Chamber	4.10	2.86	0.36	0.145	37	23.75	630
1996	Senate	4.10	3.11	0.42	0.145	40	11.60	315
2001	Chamber	3.47	2.52	0.35	0.126	33	23.75	630
2001	Senate	3.47	2.64	0.42	0.126	40	11.60	315
2006	Chamber	5.50	2.02	0.37	0.128	37	22.85	630
2006	Senate	7.64	2.03	0.37	0.128	52	15.45	315
2008	Chamber	3.79	2.74	0.39	0.160	30	22.85	630
2008	Senate	3.67	2.68	0.39	0.160	29	15.45	315
2013	Chamber	5.33	4.01	0.39	0.129	47	22.89	630
2013	Senate	5.16	3.85	0.41	0.129	58	15.75	315
2018	Chamber	5.10	3.36	0.36	0.176	28	9.08	630
2018	Senate	5.10	3.36	0.39	0.176	28	6.46	315

Table 4 Political and institutional specific variables for the Italian elections 1992-2018 (second part).

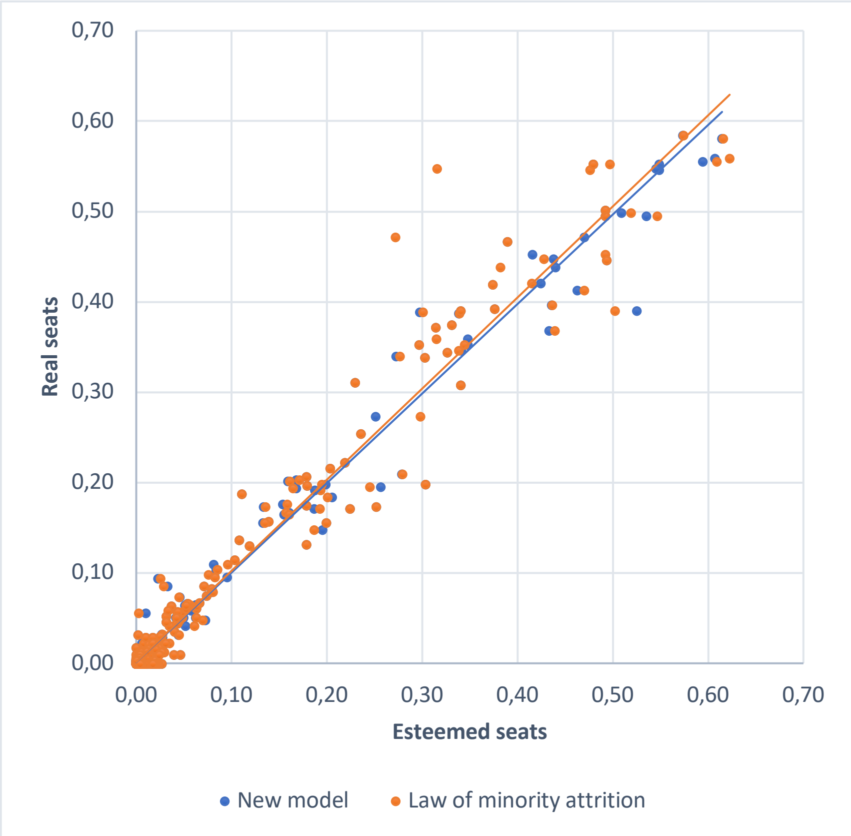
Elections	Assembly	P (Mil)	T	P1	P2	s	n	n_2
1992	Chamber	56.80	0	0.158	0.210	-	1,02	1,02
1992	Senate	56.80	0	0.150	0.202	-	1,03	1,03
1994	Chamber	56.84	0.04	0.241	0.170	0.069	2,25	2,31
1994	Senate	56.84	0	0.495	0.237	0.094	2,16	2,05
1996	Chamber	56.86	0.04	0.285	0.168	0.046	2,07	2,13
1996	Senate	56.86	0	0.506	0.226	0.089	2,10	1,99
2001	Chamber	56.97	0.04	0.303	0.181	0.039	2,54	2,59
2001	Senate	56.97	0	0.532	0.230	0.089	2,30	2,17
2006	Chamber	58.14	0.04	0.498	-	-	1,06	1,11
2006	Senate	58.14	0.08	0.498	0.497	-	1,06	1,10
2008	Chamber	58.83	0.04	0.468	-	-	1,05	1,15
2008	Senate	58.83	0.08	0.468	0.376	-	1,07	1,15
2013	Chamber	60.23	0.04	0.296	-	-	1,04	1,41
2013	Senate	60.23	0.08	0.296	0.292	-	1,05	1,34
2018	Chamber	60.60	0.03	0.337	0.204	-	1,50	1,58
2018	Senate	60.60	0.03	0.351	0.200	-	1,55	1,57

The empirical results confirm the new logical models proposed. The fitted regression is strong: the estimated seats' coefficient is near 1, significance at least at 99%, and the constant is near 0, as we would have expected, since its proximity to 0 makes its significance meaningless. The variance explained by the model is 97.63%. Using the simple law of minority attrition (using N to quantify the opponents) to estimate the real seats, the R^2 is slightly lower at 96.14%, 1.49 percentage points lower than the new model proposed.

Notwithstanding the 1.49 percentage points improvement in R^2, the Root MSE, which is the standard deviation of the codomain, is equal to $\pm$2.164 percentage points. This means that the average error of the post-diction party seats obtained using this new proposed model is equal to 2.16 percentage points; using the simple law of minority attrition, I obtain an error of $\pm$2.765, which is 0.601 percentage points or 27.8% higher (0.601/2.164). Therefore, a forecast for a party's seats equal to 2.5% $\pm$2.164% produced by my new model would be equal to 2.5 % $\pm$ 2.765% produced by the law of minority attrition. Hence, the less the party's share the more the relative error increase.

I can also proceed to calculate the adjusted R^2 (R^2Adj.) which allows assessing the impact of the massive number of the variables used. R^2Adj. $= 1 - \frac{(1-R^2)*(n-1)}{(n-k-1)}$, where n represents the dimension of the sample and k the number of variables. Being 14 the "generating" variables used: $v, S, P, M_d, M_c, T, N_{cl}, N_p, p_1, p_2, s_c, s_s, G, G_d$, k=14, n =354, the R^2 Adj. is equal to 0.975, almost equal to the pure value of 0.976. Below, in figure 9, I show the scatter plot for the expected and real values: the blue color refers to the new model proposed, whilst the orange refers to the simulation obtained applying the simple law of minority attrition.

Figure 9 Real seats on estimated seats (new model vs. law of minority attrition).



Chapter 3

Predicting seats using past information

3.1. Introduction

Until now, all models which use political and institutional variables are postdictive. Even in Predicting Party Size, the "prediction" of party sizes in a party system, given by $s_i = \frac{(1-\sum s_j)}{(N_0-i+1)^{\frac{1}{2}}}$ (2007b, p. 157), applies party shares s and N_0 in a specific temporal time. If one of the variables in the formula was substituted with one from the past (t-1), for example $s_{1,(t)} = F(s_{1,t-1} + \text{c})$, s_i would go to fully and truly predict the party shares, using s_1 in t to predict the same in t+1, and then N_0, N_2, and other s_i, in t+1.

For these reasons, I think it is helpful to study s_1 expressed in new terms concerning: the principal past variable $s_{1,t-1}$, the political terms (N_2 and N_0) and the institutional terms (MS), both indicated in the previous formula with the term c. Like a waterfall, this will imply the possibility to improve the electoral simulations, simply by substituting the future values in the law of minority attrition obtained in the previous chapter. Methodologically, I suggest adding another leg to the "two leg science" mentioned in the introduction: "how things are" and "how things should be" can be complemented by a third perspective of "how things have been". This perspective is represented by the most straightforward principle of autoregressive models (Lutkepohl H. Kratzig M., 2004) through the "memory of the system": in this case, in order to obtain s_1, the same values of it are used - for each country - in the previous election, not only considering $s_{1,t-1}$, but also considering other independent variables connected to s_1 in t, such as the number of parties N_0 and N_2, applied in the past. Furthermore, the connections between the number of parties and s_1 in t have been implemented.

In the current literature, there are purely postdictive tools given by the rules $s_1 = {N_2}^{-3/4} = {N_0}^{-1/2} = (MS)^{-1/8}$ obtainable from the four pillars[42], in particular, making the geometric mean between ${N_2}^{-3/4}$ and

[42] See also (Shugart - Taagepera, 2017, p. 106-7).

$N_0^{-1/2}$. I propose the possibility of obtaining s_1 through a new exponential postdictive function of N_0 and N_2 – parametrizing them – to better express s_1 than using the simple geometric mean previously mentioned, where R^2=62.74 for the function derivable from Taagepera et al. writings, and R^2=99.27 for my proposed one.

Nevertheless, I want to set my approach to be much more robust than a postdictive one, and I therefore introduce a predictive estimation of party seats from the previous elections using a weighted regression with independent variables jointly: 1) the product of the assembly size and the district magnitude MS, 2) the past values of the biggest party shares considered $s_{1,t-1}$, and 3) the number of Effective parties and simply considered $N_{0,t-1}$and $N_{2,t-1}$.

Here are the passages done. The first step has been to adapt the previous improved postdictive function – the exponential one – substituting the independent variables N_0 and N_2 in t-1 (to obtain s_1) in the formula, and adjusting it with new parameters, to become predictive. The second step has been to include the geometric mean between the previous result and $s_{1,t-1}$, as higher than $s_{1,t-1}$or $N_{0,t-1}$and $N_{2,t-1}$taken alone. The third step has been to obtain an additive best model that capitalizes the previous steps thanks to the pieces of knowledge on the MS product. The final result is the embedded model that considers both an additive and multiplicative (interaction) of the MS product with the previous ones, parametrizing them.

The model obtained is not just a simple time-series of s_1, because the several interactions of the previous political variables – N_2, N_0 in t-1 - and the institutional ones - M, S - minimize the variance of the regression model. This allows to "institutionalize" political variables that are acceptably stable over time like M and S, since they belong to past elections and are able to increase their explicative (and predictive) power.

In detail, to obtain a prediction, I have suggested not to only consider a straightforward geometric mean between the previous variables, for two reasons:

1) the presence of double (but complementary) heteroskedasticity – this means that plotting the previous variables, I can observe that: the higher the past value of s_1 - which is $s_{1,t-1}$ - for forecast s_1, the higher its error; inversely, the lower the MS product, the higher the forecast precision of s_1 in correspondence of higher values of s_1. This methodology provides a specific solution for politics to the problem of heteroskedasticity much more simply and asymptotically more efficient than other methods generally used in econometrics (Arellano - Bond, 1991);

2) to warrant the stability of the variables through time. Having tested $s_1$on $s_{1,t-1}$, I have noticed that the relative scatter plot gets a strange asymmetry in the fitting line, suggesting that the anchor points would be [0,0] and [1,1]. For this reason I suggest the application of the final model with the symmetric regression (Taagepera R. , 2008a, p. 154-175). This test, reapplied to the other independent variable $N_{2,t-1}$and $N_{0,t-1}$ in a similar previous final model reparametrized, did not improve the final R squared, bringing a logical inconsistency of the parameter's values respect the expectations of the "number of party" side of the equation. For this reason, I decided not to apply the symmetric regression.

This cross country analysis allows predicting s_1 using its past value, $N_{2,t-1}$ and $N_{0,t-1}$ (referred to seats), and MS, with an R^2=73.6% (R^2adj=73.3%), 19.2 percentage points better than Taagepera's and Shugart's correlation of the simple equation $\log(s_1) = \log(MS)$. Furthermore, they only consider 298 cases (Id. (p. 111-2)), making a pick-up selection of the cases in application of the more stringent Lijphart's criteria (1994) in Taagepera and Shugart (2017, p. 112, note 11). Using the same dataset (Struthers - Li - Shugart, 2018), my analysis has used the whole sample, consisting of 607 cases; this allows generalizing results as much as possible, including any limit case and political regimes.

3.2. From the basic tools to a blended model

To build the new model I start from the basics set in the introduction. Being S=M, N_0 can vary from 1 to S; $N_0 = \sqrt{1 * M} = \sqrt{1 * S}$ hence, applying the micro-mega rule, we have that $N_0 = \sqrt{\sqrt{M} * \sqrt{S}}$ then $N_0 = (M * S)^{\frac{1}{4}}$.[43]

I can now connect N_2 to N_0 in function of s_1. To do this, I introduce N_∞, logically equal to the maximum ponderation of the biggest party share s_1 such that $N_\infty = \frac{1}{(s_1)}$,[44] implying that $N_\infty < N_2 < N_0$. Therefore, I can look for $s_1$in function of N_0:

$$s_1 = \sqrt{1 * \frac{1}{N_0}} = {N_0}^{-0.5}$$

s_1 is obtained from N_0 through the geometric mean, knowing that the maximum of s_1 can be equal to 1, and its minimum is logically equal to $\frac{1}{N_0}$.[45]

Putting to system all the previous passages, as done by Taagepera (Predicting Party Sizes, 2007b, p. 138-9), we obtain: $N_\infty = \frac{1}{(s_1)} = \sqrt{N_0} \Longrightarrow N_\infty = (M * S)^{\frac{1}{8}} \Longrightarrow$

$$s_1 = (M * S)^{-\frac{1}{8}}$$

Putting all to system we obtain:

$$N_\infty = \frac{1}{(s_1)} \Longrightarrow \left(N_\infty = (M * S)^{\frac{1}{8}}\right) < N_2 < \left((M * S)^{\frac{1}{4}} = N_0\right)$$

Then, applying again the geometric mean I obtain:

$$N_2 = (M * S)^{\frac{3}{16}} \cong (M * S)^{\frac{1}{6}}$$

[43] (Taagepera R. , 2007b, p. 92,97-8,116-121)

[44] The value is obtained from the formula $N_a = [\sum(s_i)^a]^{1/(1-a)}$ (Laakso-Taagepera, 1979, p. 6-7), with $a \rightarrow \infty$.

[45] (Taagepera - Shugart R. e., 1993); (Shugart - Taagepera, 2017, p. 106)

This rounding around the statistical mode of exponents is reasonable also in consideration of the empirical test about it [46], finally obtaining the "mother" relation:

$$\mathrm{N}_{\infty}{}^{4} = \mathrm{N}_2{}^{3} = \mathrm{N}_0{}^{2} = (M * S)^{\frac{1}{2}}$$ [47]

Hence if I had to express s_1 in function of N_2, it would be sufficient to divide the exponent of the product $s_1 = (\mathrm{MS})^{k}$, in which we know k= $-\frac{1}{8}$, by the MS product expressing $\mathrm{N}_2 = (MS)^{k_1}$; I know k_1 to be equal to $\frac{1}{6}$, therefore the ratio $\frac{k}{k_1}$ indicates how many times N_2 is multiplied to obtain s_1. Then:

$$\mathrm{N}_2{}^{\frac{k}{k_1}} = s_1 \Longrightarrow \mathrm{N}_2{}^{\frac{-1}{8}:\frac{1}{6}=\frac{-1}{8}*6} = \mathrm{N}_2{}^{\frac{-3}{4}} = s_1 \Longrightarrow s_1{}^{\frac{-4}{3}} = \mathrm{N}_2$$ [48]

Coming back to finding the best postdiction of s_1 possible from N_2, N_0, the simplest formula which I can apply is the geometric mean between the logical links founded by Taagepera in his publications, as follows:

$$s_1 Taagepera|(\mathrm{N}_0, \mathrm{N}_2) = \sqrt{N_2{}^{-3/4} * N_0{}^{-1/2}}$$

From here I will be using the expression "y|(x)" - as in the formula above - to express y conditional to x or, in other words, y in function of x. The previous formula means $s_1 Taagepera$ in function of N_0 and N_2.
To test this and the following hypotheses introduced, I will be managing the same dataset used in Taagepera and Shugart (2017) on a nationwide basis - concerning countries - for 974 elections (Struthers - Li - Shugart, 2018). I noticed 10 cases for which N2≤N0; this should be logically impossible (because of the opposite relation introduced before) and I have therefore cleaned the dataset from those; it probably happened due to the different data collection sources.
I obtain the following figure 10:

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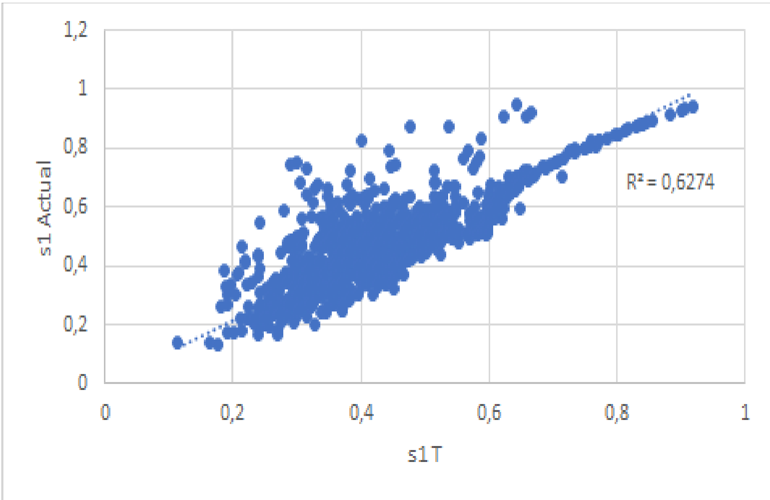
[46] *(Taagepera R. , 2007b, p. 153)* tested on 25 country dates by Liparth.
[47] For a wide discussion, see *(Taagepera R. , 2007b, p. 97;154-156;226).*
[48] The result is reported by Shugart and Taagepera *(2017, p. 107)* even though through a different equation.

Figure 10 s_1 in function of N_2 and N_0- on Taagepera assumptions - through the geometric mean.



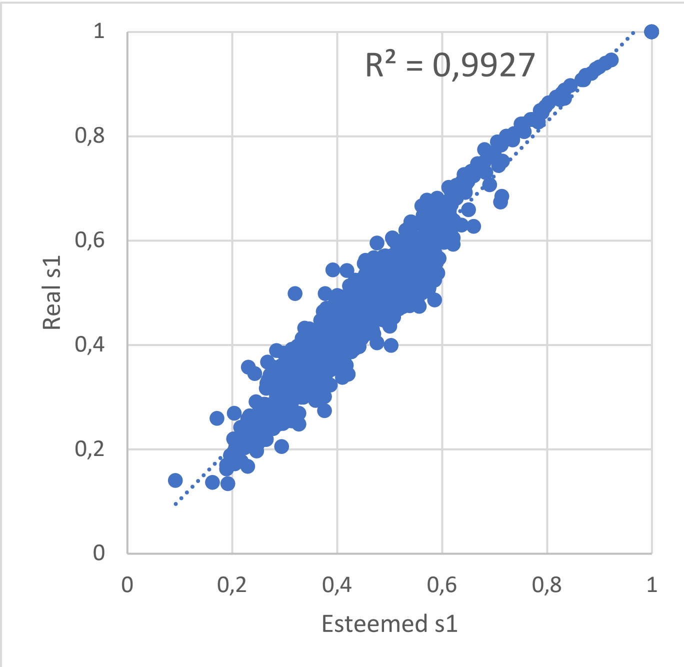
I can impose a further condition, for which $1 \geq \frac{1}{N_2} \geq \frac{1}{N_0} \Longrightarrow 1 \geq HH \geq \frac{1}{N_0}$, deriving from $N_0 \geq N_2 \geq 1$.

At this point, I propose to create from scratch some other relations to test. I can re-start from $s_1 = N_2^{\frac{-3}{4}} \Longrightarrow s_1 = \frac{1}{N_2^{\alpha_1}}$, then firstly adjust the parameter $\frac{3}{4}$ empirically. Furthermore, the ratio $\frac{N_2}{N_0}$ defined in the interval [0,1] could impact the previous formula that expresses s1, because in case this ratio tends to 1, all the parties get the same share then: $\frac{1}{N_2^{\alpha_1}} \rightarrow \frac{1}{N_2}$. On the other hand, for $\frac{N_2}{N_0} \rightarrow 0$, this would mean that there is too much fragmentation - and then outliers - implying a rebalancing of the ratio $\frac{1}{N_2}$, aiming to obtain higher values of this simplest ratio; to correct this, I elevate it to the power of $\frac{N_2}{N_0}$. Finally, I have introduced some parametrization for each variable through α_1, α_2, and α_3, intending to obtain further refinement:

$$s_1|(N_2, N_0) = \left(\frac{1}{N_2^{\alpha_1}}\right)^{\frac{N_2^{\alpha_2}}{N_0^{\alpha_3}}} = \left(\frac{1}{N_2^{0.86}}\right)^{\frac{N_2^{0.05}}{N_0^{0.1}}}$$

The parameters introduced until now are empirically founded through the maximum likelihood estimation method, just like the next ones. The scatter plot below, the figure 11, confirms empirically that the logical model just built can explain s_1 in function of N_2 and N_0 (in a postdictive way (t)).

Figure 11 The real s_1 on estimated s_1, through the new function of N_2 and N_0.



Even though the simple R^2 shown is equal to 99.27%, the R^2adj. is lower, equal to 92.1%, having several parameters; it is in any case higher than the previous "Taageperian" geometric mean, than can be considered the "winning" one.

At this stage, I can cast into system all the pieces of knowledge, translating these from postdictive to predictive (Taagepera R. , Predicting Party Sizes, 2007). I can take advantage of the Taageperian theory of "two-leg science" (Taagepera R. , 2017, p. 7-11), not just as an epistemological science's foundation, but for how it can be used to build the model and to test it empirically. This offers proof that this approach is valid both for methodological, and empirical and actual applications of how it can work better than unidirectional approaches.

Taagepera's methodology foresees that scientific theories must consider two sides: 1) how things are, and 2) how these should be. This assumption is related to the fact that we cannot just use an inductivist approach to a problem's resolution because the "science does not start from the observation and the induction does not exist" (Antiseri, 2007, p. 5-19), because this would mean applying some methodologies in a mechanic way without any control. As the epistemologist Kuhn (id. (p. 255-258)) would have said, doing "normal science" excluding any innovation from the start, surely it can work to solve some well-codified science problems; however, this may represent an abstract way of doing science when we have an unsolved problem or try to enlarge the scope of the science.

Moreover, when graphing data some clues could be observed such as some strange shapes, distribution, anchor points and forbidden areas; generally, this happens when we know something in advance that obligates the variables to exist in a specific interval and/or in a specific shape. Finally, the existence of specific correlation forms are widely under-evaluated among those variables; for instance, the simple multiplication of variables called covariate can be replaced by a specific mathematical function of correlation between the same variables.

3.3.Final model: introducing past information into the blended model

In the specific case of s_1 prediction, I suggest adding another leg to "how things are" and "how things should be", introducing the perspective of "how things have been". This perspective concretizes the simplest principle of autoregressive models (Lutkepohl H. Kratzig M., 2004) to check whether the previous s_1 values of the previous election $(s_{1,t-1})$ in the same country, if this exists, can express the current ones, as follow:

$$s_1|s_{1,t-1} = s_{1,t-1}$$

In the same way, we can re-propose the postdictive formula $s_1|(N_2, N_0)$, applying that to the past values $(N_2, N_0)_{t-1}$ of N_2, N_0. This represents another side, showing how things should be on the logical ground because of the logical properties introduced constructing the $s_1|(N_2, N_0)$ index.

$$s_1|(N_2, N_0)_{t-1} = \left(\frac{1}{{N_{2,t-1}}^{\gamma_1}}\right)^{\frac{{N_{2,t-1}}^{\gamma_2}}{{N_{0,t-1}}^{\gamma_3}}} = \left(\frac{1}{{N_{2,t-1}}^{1.06}}\right)^{\frac{{N_{2,t-1}}^{0.08}}{{N_{0,t-1}}^{0.06}}}$$

I now test whether the geometric mean of the formula of "how things have been" with "how things should be", yields a stronger prediction of the simple ones considered singularly.

$$s_1|(s_1, N_2, N_0)_{t-1} = \sqrt{s_{1,t-1} * \left(\frac{1}{{N_{2,t-1}}^{1.06}}\right)^{\frac{{N_{2,t-1}}^{0.08}}{{N_{0,t-1}}^{0.06}}}}$$

The empirical tests below - Table 5 - show that $s_1|(s_1, N_2, N_0)_{t-1}$ effectively gets a more substantial predictive power of the simple MS product, as reported in Shugart and Taagepera (2017, p. 112): the former obtains an R^2 adj. of 63.6% (in model 4), whilst the latter obtains an R^2 (not adjusted) of 54.4%. Furthermore, it widens the sample to include all cases available in the dataset, without making a pick-up selection of cases (ibid., note 11) in the application of the Lijphart's criteria (1994). This also allows generalizing results as much as possible, to include any limit case[49].

[49]Because of this case selection, the present R^2 is equal to 33.2% instead of the Shugart and Taagepera's 54.4% *(2017, p. 112)*; however, the present sample is 755 cases, respect their sample count of 298.

The higher strength of the methodology behind $s_1|(s_1, N_2, N_0)_{t-1}$ respect MS product, could also be explained through the Fermi's Piano Tuners estimation experiment (Taagepera R. , 2015, p. 215-219), which reasonably approximates how many piano tuners are there in the city of New York (id. (p. 216)) thanks to the connections among connections of variables, producing an interlocking relationship (id. (p. 89-95)).

Table 5 Correlation models of s_1 in function of MS product, $s_{1,t-1}$, $N_{2,t-1}$ and $N_{0,t-1}$

VARIABLES	(Model 1) Log(s1)	(Model 2) Log(s1)	(Model 3) Log(s1)	(Model 4) Log(s1)	(Model 5) Log(s1)	(Model 6) Log(s1)
Log $(s_1|s_{1,t-1})$		0.761***			0.676***	
		(0.0220)			(0.0262)	
Log $(s_1|(N_2, N_0)_{t-1})$			0.257***			
			(0.00706)			
$Log(s_1|(s_1, N_2, N_0)_{t-1})$				0.305***		0.283***
				(0.00824)		(0.0102)
Log (MS)	0.111***				0.0373***	0.0310***
	(0.00577)				(0.00509)	(0.00521)
Constant	0.0356**	0.0893***	0.0593***	0.0567***	-0.0132	0.00797
	(0.0168)	(0.00827)	(0.00871)	(0.00864)	(0.0125)	(0.0125)
Observations	755	874	785	785	681	614
R-squared	0.332	0.578	0.629	0.637	0.663	0.713
n° independent variables and parameters (excluded constants)	2	1	2	3	3	5
R-squared adj.	0.330	0.578	0.628	0.636	0.662	0.711

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Figure 12 s_1 in function of logged $(s_1, N_2, N_0)_{t-1}$.

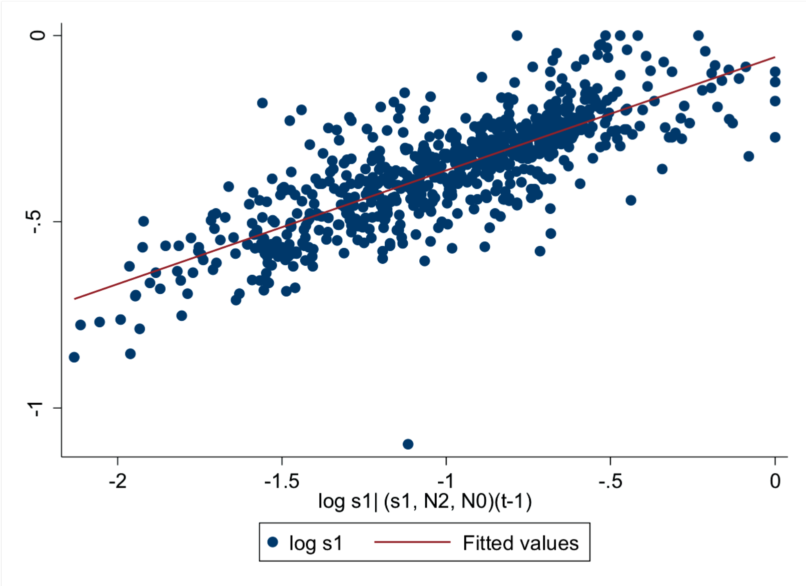
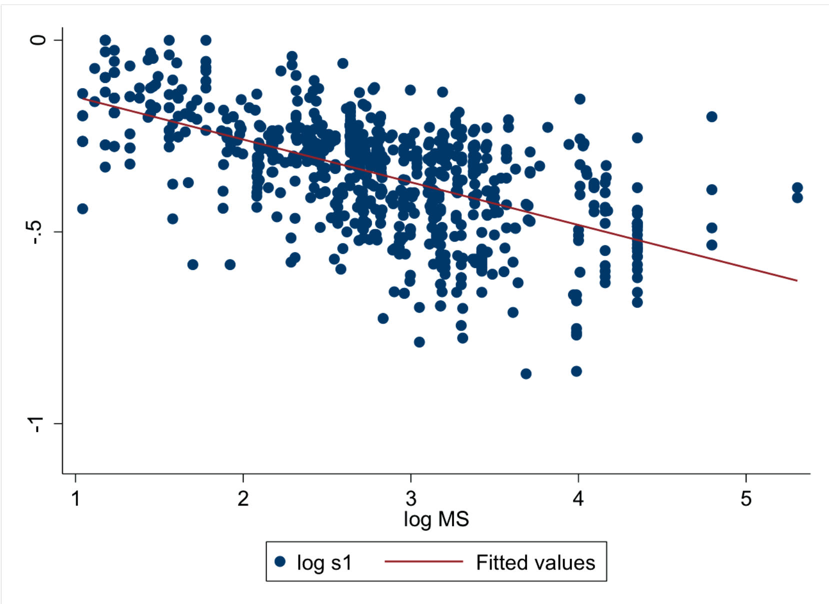
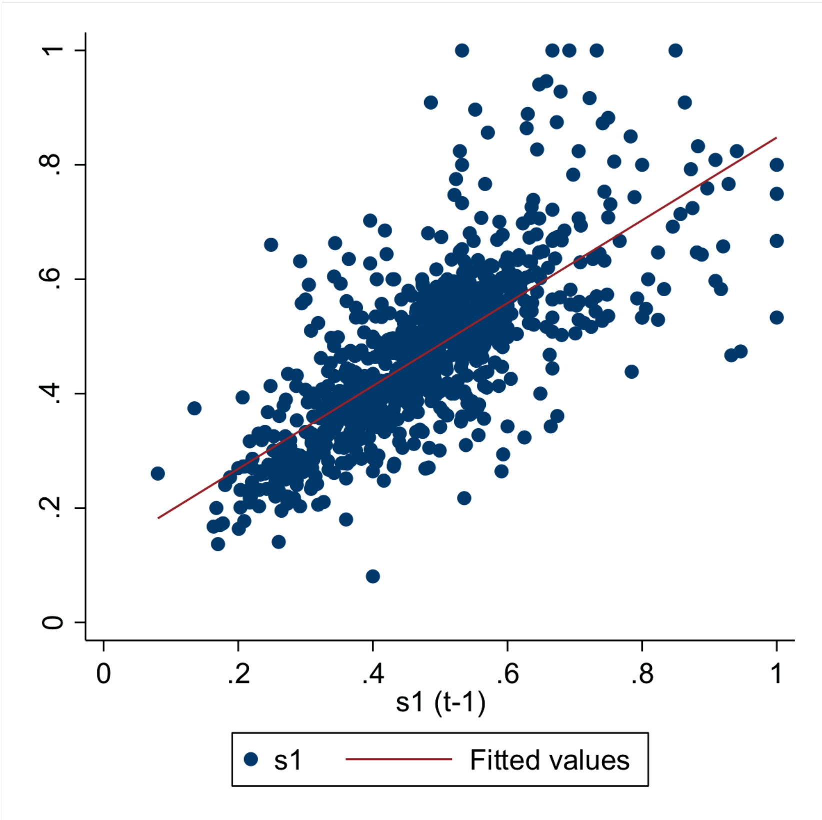


Figure 13 s_1 in function of logged MS product.



The figure 13 above shows that the lower the MS product, the higher the precision in forecasting s_1, which corresponds to higher values of s_1. Inversely, the figure 14 below shows that the higher $s_{1,t-1}$, the higher the error in forecasting s_1:

Figure 14 s_1 in function of logged $s_{1,t-1}$.



This highlights a problem of complementary heteroskedasticity in predicting s_1. Therefore, I propose a new model to overcome such problem, blending MS and $(s_1)_{t-1}$ to more accurately predict s_1 in function of those.

It is a good moment to clarify the use of logarithmic expressions and functions in these models. Beta coefficients used in basic regressions are simple variable impact weights. When models include covariates, logarithms are needed to determine weights of covariate's variables, preventatively calculated individually through logarithmic equations. For this reason, both MS, $(\mathrm{s}_1)_{t-1}$, and later N_2, N_0, and all related covariates, will be logged when appropriate.

To develop my blended model, I multiply each variable for its respective logical weights, defined in function of the accuracy of the prediction returned by the opposite variable.

I start with $MS, (\mathrm{s}_1)_{t-1}$ only, and later expand to include $(\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1}$ in order to confirm the added value of the two-leg regressor $(\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1}$.

$$\begin{aligned} &\log(\mathrm{s}_1)|MS, (\mathrm{s}_1)_{t-1} \\ &= \beta_1 \log(MS) * \log(\mathrm{s}_1|(\mathrm{s}_1)_{t-1}) \\ &+ \beta_2 \log(\mathrm{s}_1|(\mathrm{s}_1)_{t-1})(1 - MS^{-0.111})^{1.2} \end{aligned}$$

I also introduce a covariate of covariate variable:

$$\mathrm{Covs} = log(MS) * \log(\mathrm{s}_1|(\mathrm{s}_1)_{t-1}) * log(\mathrm{s}_1|(\mathrm{s}_1)_{t-1})(1 - MS^{-0.111})^{1.2}$$

Thus obtaining:

$$\begin{aligned} &\log(\mathrm{s}_1)|MS, (\mathrm{s}_1)_{t-1} = \\ &\beta_1 log(MS) * \log(\mathrm{s}_1|(\mathrm{s}_1)_{t-1}) + \beta_2 log(\mathrm{s}_1|(\mathrm{s}_1)_{t-1})(1 - MS^{-0.111})^{1.2} \\ &+ \beta_3 covs \end{aligned}$$

$$\begin{aligned} &\log(\mathrm{s}_1)|MS, (\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1} = \\ &\beta_4 log(MS) * \log(\mathrm{s}_1|(\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1}) \\ &+ \beta_5 log(\mathrm{s}_1|(\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1})(1 - MS^{-0.111})^{1.2} \end{aligned}$$

It is now possible to introduce another covariate variable:

$$\begin{aligned} \mathrm{Covf} = {} & log(MS) * \log(\mathrm{s}_1|(\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1}) \\ & * log(\mathrm{s}_1|(\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1})(1 - MS^{-0.111})^{1.2} \end{aligned}$$

$$\begin{aligned} &\log(\mathrm{s}_1)|MS, (s_1, N_2, N_0)_{t-1} = \\ &\beta_4 log(MS) * \log(s_1|(\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1}) \\ &+ \beta_5 log(\mathrm{s}_1|(\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1})(1 - MS^{-0.111})^{1.2} + \beta_6 covf \end{aligned}$$

Table 6 below shows the tests of these further models.

Table 6 Correlation models of s_1 in function of MS product, $s_{1,t-1}$, $N_{2,t-1}$ and $N_{0,t-1}$, logged and not

VARIABLES	(Model 7) Log(s1)	(Model 8) Log(s1)	(Model 9) Log(s1)	(Model 10) Log(s1)
$Log(MS) * log(s_1 \vert (s_1, N_2, N_0)_{t-1})$			-0.0836***	-0.0991***
			(0.0108)	(0.0130)
$Log(s_1 \vert (s_1, N_2, N_0)_{t-1}) * (1 - MS^{-0.111})^{1.2}$			1.835***	1.973***
			(0.171)	(0.183)
covf				-0.00646**
				(0.00302)
$log(MS) * \log(s_1 \vert (s_1)_{t-1})$	-0.0795***	-0.0773***		
	(0.0151)	(0.0130)		
$log(s_1 \vert (s_1)_{t-1})(1 - MS^{-0.111})^{1.2}$	1.818***	1.800***		
	(0.214)	(0.204)		
covs	-0.00115			
	(0.00406)			
Constant	-0.112***	-0.110***	-0.0849***	-0.101***
	(0.0104)	(0.00772)	(0.00802)	(0.0108)
Observations	672	672	607	607
R-squared	0.689	0.689	0.734	0.736
n° variables and parameters (excluded constants)	5	4	6	7
R-squared adj.	0.687	0.687	0.731	0.733

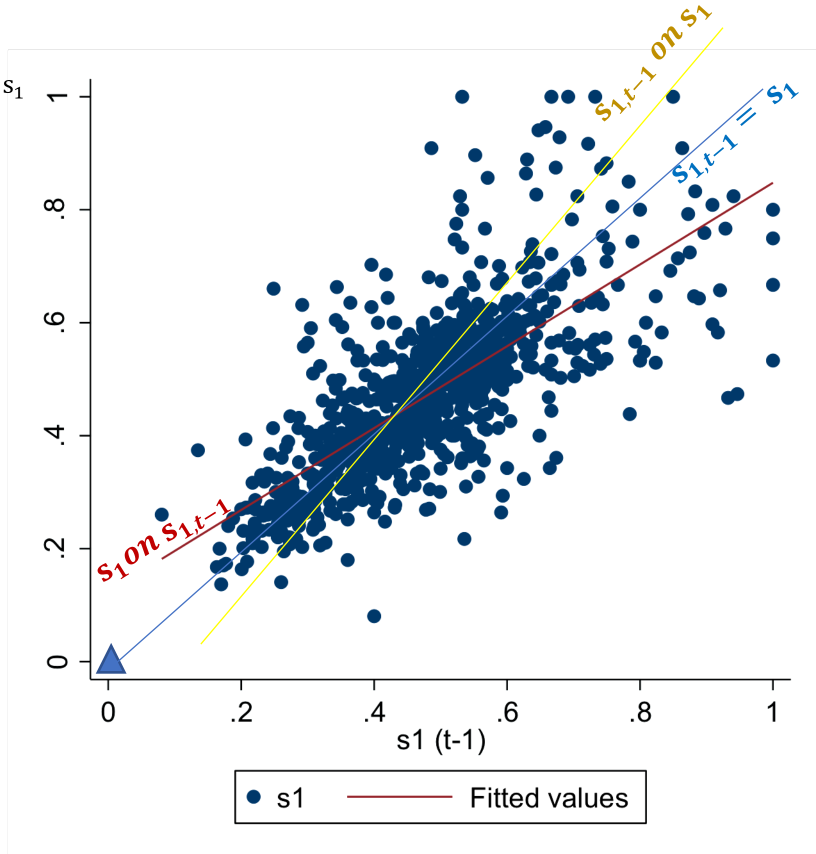
Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1
β coefficients before the asterisks

I apply a counter proof to theoretically and empirically confirm the strength of the simple geometric mean of the two-leg regressor used to build the model $s_1|(s_1, N_2, N_0)_{t-1}$. We can empirically test whether a parametrization of both $(s_1|(s_1)_{t-1})$ and $(s_1|(N_2, N_0)_{t-1})$ allows the scatters to fit the anchor points [0,0] and [1,1] respectively, and correct the asymmetries shown in the graph below – figure 15 - as much as possible. This produces a better model (number 10, considering the $R^2\ adj.$), and helps me verify whether the parametrization is consistent with the logical expected one.

I notice an odd asymmetry in the fitting line of the s_1 on $s_{1,t-1}$ graph reported above: the logical expectation of the scatter's fitting line would present the anchor points of [0,0] and [1,1], however the actual line is rotated above the anchor point [0,0] hinged on the barycenter which I am introducing below.

Hence I apply the tool of symmetric regression (Taagepera R. , 2008a, p. 154-175) to the original fitting line, obtaining $s_{1,t-1} = s_1$:

Figure 15 The s_1 symmetric regression lines $s_{1,t-1} = s_1$, obtained from s_1 on $s_{1,t-1}$ and $s_{1,t-1}$ on s_1.



The line resulting from this application has the property to pass across the intersection of s_1 on $s_{1,t-1}$ and $s_{1,t-1}$ on s_1. Its angular coefficient is defined as $B = \pm(bb'')^2$.[50] The b'' value is determined as the inverse of

[50] Cfr. *(Taagepera R. , 2008a, p. 163).*

the angular coefficient b', belonging to the regression line of $s_{1,t-1}$ on s_1, and the angular coefficient b, belonging to the regression line of s_1 on $s_{1,t-1}$ (which is the actual regression line).

Table 7 Regression's coefficients s_1 on $s_{1,t.1}$ and $s_{1,t-1}$ on s_1, building the symmetric regression $s_{1,t-1} = s_1$

VARIABLES	(15) s_1	Empirical $s_{1,t-1}$	Empirical s_1
$s_{1,t-1}$ (b)	0.725***		
	(0.0228)		
$s_1(b')$		0.742***	
		(0.0233)	
$s_{1,t-1}\left(b'' = {}^{1}/_{b'}\right)$			${}^{1}/_{0.742} = 1.35$
Constant	0.123***	0.128***	${}^{-0.128}/_{0.742} = -0.173$
	(0.0113)	(0.0114)	
Observations	874	874	
R-squared	0.537	0.537	

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

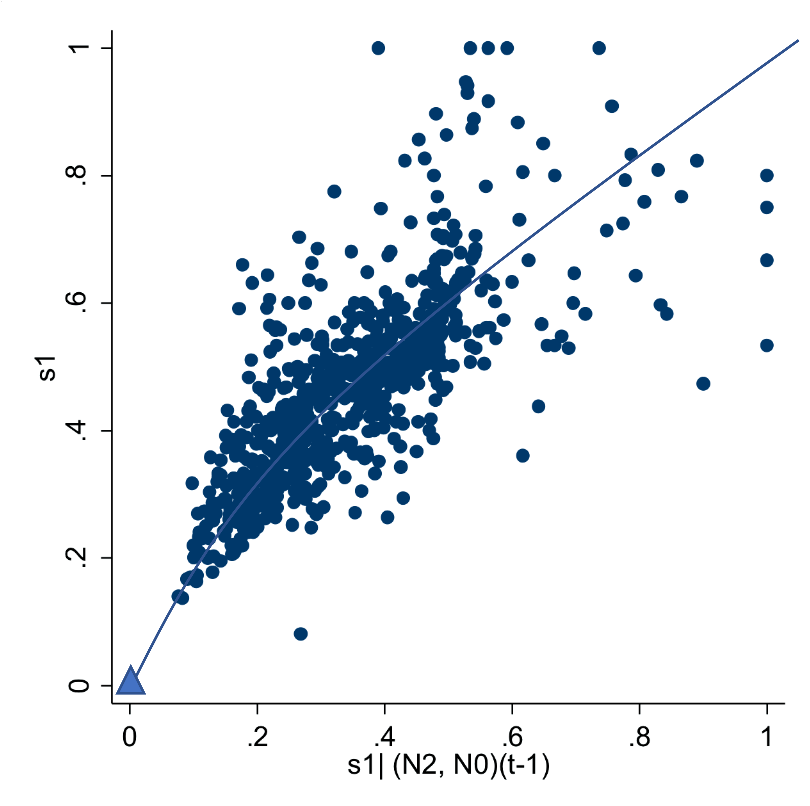
Hence, the angular coefficient of the symmetric regression $B = \pm\sqrt{bb''}$ is equal to $B = \pm\sqrt{1.35 * 0.725} = 0.989$. I can calculate the ratio between B and 1 (here logically determined), as the logical multiplicative factor of discard of the model, $\frac{Actual}{\text{Estimated(logically)}} = Discard\ Factor$ (Cfr. (Taagepera R. , 2007, p. 168, 288; 2015, p. 38-9), $\frac{0.989}{1} = 0.989$, implying that the Actual value B is off by a factor of ×÷ 1.1 percentage points (multiplied or divided by 1.1 percentage points).

To determine the constant (A) which matches B as follows: $s_1 = Bs_{1,t-1} + A$, I cast into system the two specular regression lines with the aim to identify their intersection point and superimpose the passage across that, as follows:

$$s_1 = 0.725s_{1,t-1} + 0.123 = 1.35s_{1,t-1} - 0.173 \Longrightarrow 0.625s_{1,t-1} = 0.296 \Longrightarrow$$
$$s_{1,t-1} = 0.474 \Longrightarrow s_1 = 1.35 * 0.474 - 0.173 = 0.467 \Longrightarrow$$
$$s_1 = Bs_{1,t-1} + A \Longrightarrow 0.467 = 0.989 * 0.474 + A \Longrightarrow A = 0.00179 \Longrightarrow$$
$$s_{1,sim} = 0.989s_{1,t-1} + 0.00179$$

On the other hand, s_1 on $(N_2, N_0)_{t-1}$ presents some exponential asymmetry with an expected exponent <1 (Taagepera R. , 2008a, p. 95-102), as suggested by the graph below, the figure 16:

Figure 16 s_1 estimated in function of $N_{2,t-1}$ and $N_{0,t-1}$.



Looking at $(s_1|(s_1)_{t-1})$, I considered the refitting for a symmetric regression (Cfr. (Taagepera R. , 2008a, p. 154-175; 2015, p. 142-148)) obtaining a straight line, and then a (hypothetical) better fitting of the scatter throughout the anchor points [0,0] and [1,1], thus obtaining $\left(s_1|(s_1)_{t-1,sim}\right) = 0.989s_{1,t-1} + 0.00179$.

To resolve the exponential asymmetry in $(s_1|(N_2, N_0)_{t-1})$, I can apply the simplest form of parametrisation x^{ω} since the biggest curvature of the scatter is near the origin [0,0] (Taagepera (2008a, p. 97-99; 2015, p. 66-82)), allowing the scatter to also fit the anchor points [0,0] and [1,1] (blue lines in the previous respective graphs), thus obtaining:

$$s_1|(N_2, N_0)_{t-1}^{\omega} = \left(\frac{1}{{N_{2,t-1}}^{1.06}}\right)^{\left(\frac{{N_{2,t-1}}^{0.08}}{{N_{0,t-1}}^{0.06}}\right)^{\omega}} \quad ;$$

$$s_1|\left(s_{1,sim}, (N_2, N_0)^{\omega}\right)_{t-1} = \sqrt{(0.989 s_{1,t-1} + 0.00179) * \left(\frac{1}{{N_{2,t-1}}^{1.06}}\right)^{\left(\frac{{N_{2,t-1}}^{0.08}}{{N_{0,t-1}}^{0.06}}\right)^{\omega}}}$$

Nevertheless, to obtain the final formula I apply a maximization of the R^2 through ω – because it is the only free parameter I can manipulate; $\left(s_1|(s_1)_{t-1,sim}\right)$ has to respect the angular coefficient and the constant to hit the anchor points shown before.

$$s_1|\left(s_{1,sim}, (N_2, N_0)^{\omega}\right)_{t-1} = \sqrt{(0.989 s_{1,t-1} + 0.00179) * \left(\frac{1}{{N_{2,t-1}}^{1.06}}\right)^{\left(\frac{{N_{2,t-1}}^{0.08}}{{N_{0,t-1}}^{0.06}}\right)^{1.15}}}$$

With surprise, the parameter ω obtained is 1.15, greater than 1, and not less than 1, as I was expecting from the graph, therefore logically inconsistent. Furthermore, the $R^2\,adj.$ has not reported any improvements (-0.1 %), as shown in Table 8 below.

Table 8 Comparison between the best model of s_1 in function of MS product, $s_{1,t-1}, N_{2,t-1}$ and $N_{0,t-1}$ (logged and not), and the symmetric regression of the same

VARIABLES	(Model 10) Log(s1)	(Model 11) Symmetric + parametrisation Log(s1)
$Log(MS) * log(\mathrm{s}_1 \vert (\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1})$	-0.0991***	
	(0.0130)	
$Log(\mathrm{s}_1 \vert (\mathrm{s}_1, \mathrm{N}_2, \mathrm{N}_0)_{t-1}) * (1 - MS^{-0.111})^{1.2}$	1.973***	
	(0.183)	
covf	-0.00646**	
	(0.00302)	
$Log(MS) * log\left(\mathrm{s}_1 \middle\vert \left(\mathrm{s}_{1,\mathrm{sim}}, (\mathrm{N}_2, \mathrm{N}_0)^{\omega}\right)_{t-1}\right)$		-0.0918***
		(0.0120)
$log\left(\mathrm{s}_1 \middle\vert \left(\mathrm{s}_{1,\mathrm{sim}}, (\mathrm{N}_2, \mathrm{N}_0)^{\omega}\right)_{t-1}\right) * (1 - MS^{-0.111})^{1.2}$		1.824***
		(0.168)
Covf (sim, ω)		-0.00556**
		(0.00256)
Constant	-0.101***	-0.0996***
	(0.0108)	(0.0109)
Observations	607	607
R-squared	0.736	0.736
n° variables and parameters (excluded constants)	7	8
R-squared adj.	0.733	0.732

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1
β coefficients before the asterisks

For these reasons, I choose number 10 as the best model.

Below, I show the scatter plots - the figures 15 and 16 - representing: 1) the real on fitted values logged, 2) the real values, not logged, on fitted values, logged, and 3) the real and the fitted values, both not logged.

Figure 17 Scatter plot of the real on fitted values of logged s_1, of the best model of s_1 in the function of MS product, $s_{1,t-1}, N_{2,t-1}$ and $N_{0,t-1}$, logged and not.

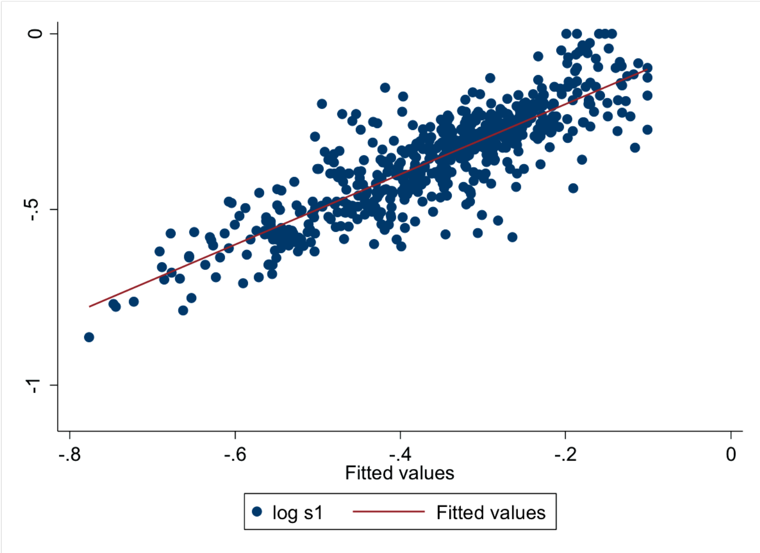
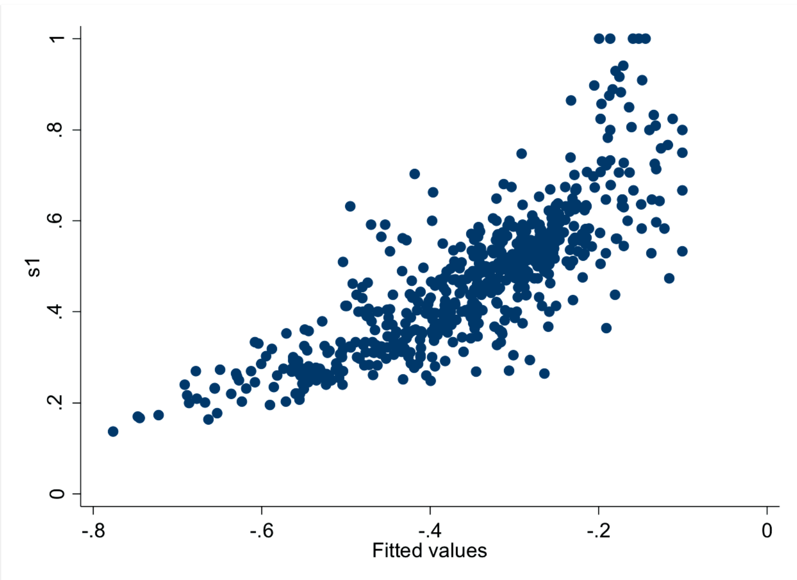


Figure 18 Scatter plot of the real s_1 on fitted values of $\log(s_1)$, of the best model of s_1 in the function of MS product, $s_{1,t-1}, N_{2,t-1}$ and $N_{0,t-1}$, logged and not.

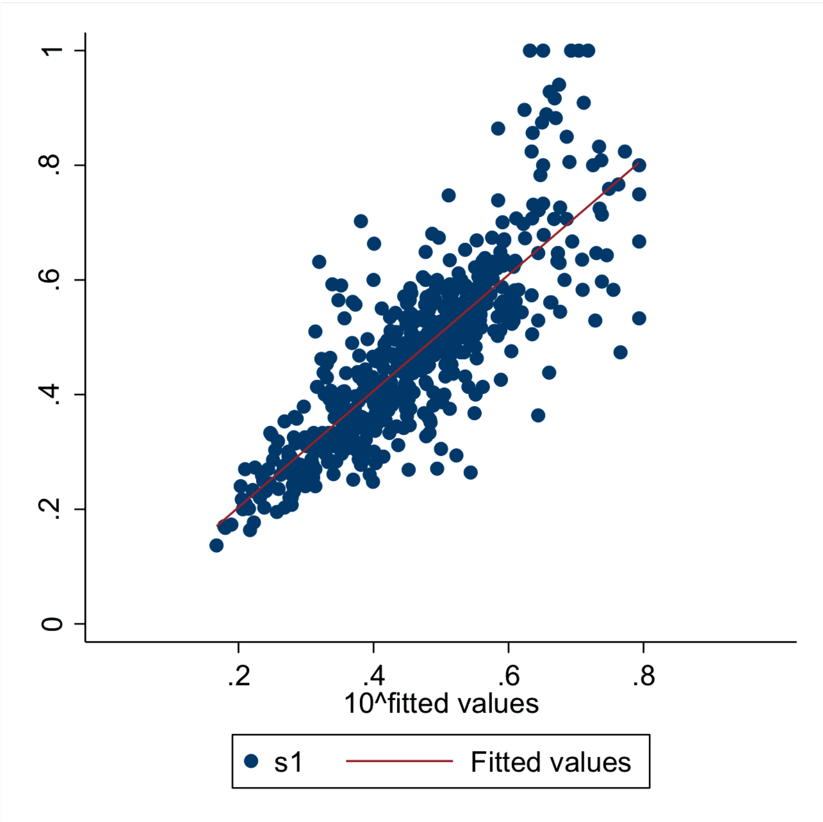


In order to obtain s_1 as the dependent variable, I must apply the definition of logarithm, since the variable is currently expressed in logarithmic terms. Knowing that the logarithm is the exponent to give the base in order to obtain the argument, I can obtain s_1 from the complex log model:

$$log(s_1) = Fittedvalues(modelx) \Longrightarrow s_1 = 10^{Fittedvalues(modelx)}$$

As represented in the following graph, the figure 19:

Figure 19 Scatter plot of the real on fitted values of s_1, of the best model of s_1 in the function of MS product, $s_{1,t-1}$, $N_{2,t-1}$ and $N_{0,t-1}$, logged and not.



3.4.Further perspectives

Widening the considerations made in this chapter, a practical future application is a machine-learning methodology that goes to minimize the variance of the regression's model, allowing to "institutionalize" political variables, because these are stable over time. I propose to take into consideration not only the past values of the variables themselves, but each political variable's average considering the previous values belonging to at least 2-3 previous elections and/or at least 10 years from the value to be predicted. The specific rule to adopt is found empirically, such that these values have gone on to produce variations of their mean values lower than 5% (similarly to a confidence interval) through time.

This chapter theorizes and formalizes the connections among s_1, $s_{1,t-1}$, $N_{2,t-1}$, $N_{0,t-1}$ and MS, supporting a more accurate description and prediction of the party system at both national and district levels. Nevertheless, as said in the introduction, the base research provides tools - as in this case - which could still have unknown applications. What has emerged from this paper could be applied to: a simple description of a party system, the impact on the Gallagher dis-representation D_2, the block Thresholds T, or the Cabinet duration C (Taagepera R. , 2007) (Shugart - Taagepera, 2017). These could have an impact on finding optimal indexes of disproportionality n_2 and the optimal values for M and S. Moreover, the learnings from this paper could be applied to make predictions of parties using historical data, which remains still unexplored for each party's share. Another suggested use could be to apply the function at national level to circumscriptions or district level in order to predict the apportionment of the national result in each district.

PART II USING LQM IN POLITICAL SCIENCE

Chapter 4

Statistical function for the parties' distribution and allotment

4.1. Introduction

When I speak about statistical distribution in general, the mind goes to the gaussian shape. If we think of the party shares, the repetition of similar party shares in a party system configures a specific frequency, which is usually represented by a gaussian curve; however, this poses a limit since gaussians curves exist in the interval [-∞,+∞], whereas party shares are defined in the interval [0,1].

This would require a different distribution curve, such as the Eulerian Beta distribution. However, like the gaussian, this distribution over-imposes a unimodal[51] shape that could be a good approximation for how many parties I will come across in correspondence of a possible party share, but this is just a mathematical approximation.

To overcome these limitations, starting from the political variables seen until now, it would be interesting to obtain a continuous function able to catch a prevailing party above the average or a group of small parties below it. For this purpose, I have applied a new theoretical function to the respective party systems of sixteen countries, revealing how the distributions of several countries' party shares do not follow a unimodal shape.

This is an advancement in the base statistical research, but it can have possible political applications. For example, taking districts into considerations, and substituting their N and N_0, this new function can output the number of seats allotted to each party at the district's level, enriching the composite indicators that describe the district's dimension in the intraparty dimension of representation, as done until now (Cfr. (Shugart - Taagepera, 2017, p. 236-258)). Another application of this function can be to evaluate the asymmetry among the parties' shares and then allow the calculus' implementation of the cabinet's life, improving the current R-squared between the actual and the predicted cabinet's life.

[51] A function is unimodal if it is monotone (then being the second derivative positive or negative, the curve appears respectively bent toward the up or down), increasing up to a certain point (the mode), and subsequently, it is monotone decreasing.

In practical terms, in order to obtain the proposed probability density function, I derive a cumulative function, obtained from the sigmoid law of minority attrition (introduced when formalizing the correlation between seats and votes), in which the exponent will be another function of N and N_0. I have calculated a graphical barycentre E, which is the expected value of the average party share, which s_1 and the group of minor parties are equidistant from, if these are distinguishable by E. I am introducing the Field and Grofman standard deviation (2007, p. 105), given N and N_0, as a new tool supporting the build of the new statistical function.

The result obtained will be a density probability function with two-to-five inflection points, defined by the following variables:

> *1)* s, which is the independent variable represented by all possible percentage of seats that can be allotted (defined from 0 to 1 (100%));
>
> *2)* f_{der} (in the final aggregated formulation), which is the dependent variable, indicating how frequent that allotment is for the respective s (defined from 0 to infinite);
>
> *3)* the redefined constant n, which is itself a function of N and N_0.

For example, if three parties get the same percentage of seats allotted at the national level, then I have three parties each with $33{,}\bar{3}\%$ of seats; therefore, the cumulative function of f_{der}, which I call f_{cum}, must be: f_{cum} equal to 0 for $0<s<33{,}\bar{3}\%$, equal to 1 for $33{,}\bar{3}\%<s<1$, and equal to 0.5 for $s= 33{,}\bar{3}\%$.

4.2. Party system fragmentation: from the literature to the new perspective.

In addition to the HH, F and N formulations above, I introduce a new relation by Feld and Grofman (2007, p. 105) to analytically describe the party system, putting a given variance σ^2 in relation with the nominal number of party N_0 and HH (therefore N_2).[52] Their model can be summarized as follows:

[52] In the original paper, N_0 appears named as the variable n, and N_2 is introduced through the HH index.

- $\mathrm{HH} = \mu + n\sigma^2$; $\mathrm{N}_2 = \frac{1}{\mu+\mathrm{N}_0\,\sigma^2}$;
- $\mu = \frac{1}{\mathrm{N}_0}$ [0,1]
- σ^2 [0,0.25] $\Rightarrow$ σ [0,0.5]

Starting from this point, it is helpful to estimate the expected value E for the biggest party share s_1. It will be sufficient to take the formula of s_1 expressed in function of N and N_0 (both in t) from the previous chapter, potentially substituting the s_1 prediction, when needed.

I calculate a graphical barycenter E, which both the smaller parties and s_1 would be equidistant from. On average, E represents the point of the ideal-typical party share of a given party system; it needs to be defined through a more complex approach than just the reciprocal of N, since the standard deviation, and then, the asymmetries need to also be taken into account. For example, on one hand, in a system characterized by more numerous small parties (as small as to be not relevant) than bigger ones, E switches towards s_1 first, then along the x-axis of the parties' shares existence [0,1] towards the right. On the other hand, with an infinite number of parties ($\mathrm{N}_0 \to \infty$), the center of gravity E would logically tend to translate to the left since the more parties, the less the expected value of the party shares of the party system; with more parties to distribute the consensus among, they will each get fewer and same votes, therefore the $\lim_{\mathrm{N}_0\to\infty} E = \frac{1}{\mathrm{N}_0} = 0$.

I formalize the limits and noteworthy points both logically founded, with a unique and comprehensive function which I can be sure will satisfy them, where E is the independent variable, and with the following anchor points:

1) $\lim_{\mathrm{N}_0\to\infty} E = 0$
2) $\lim_{\mathrm{N}_0=1} E = 1$, and any value of N_2 becomes irrelevant
3) $\lim_{\mathrm{N}_0=2} E = 0.5$, and any value of N_2 becomes irrelevant
4) the higher N_0, the most E tends to $\frac{1}{\mathrm{N}_2}$ because of the outliers' presence; conversely, the lower N_0, the most E tends to $\frac{1}{\mathrm{N}_0}$

This is the proposed function:

$$E = \frac{1}{\mathrm{N}_0}\left(\frac{\mathrm{N}_2}{\mathrm{N}_0}\right)^{\frac{\mathrm{N}_0-2}{\mathrm{N}_0}} + \frac{1}{\mathrm{N}_2}\left(1 - \left(\frac{\mathrm{N}_2}{\mathrm{N}_0}\right)^{\frac{\mathrm{N}_0-2}{\mathrm{N}_0}}\right)$$

I have imagined two monomers, which compose E from two limit points N_0 and N_2, considered in their reciprocal values to produce parties' shares. In addition, in compliance with point number 4 above, I have introduced other elements as "activation compensative weights", so that the higher N_0, the more E will tend to N_2; knowing that N_0 is always greater than N_2, $\frac{\mathrm{N}}{\mathrm{N}_0}$ exists between 0 and 1, resulting in as a simple reasonable weight for $\frac{1}{\mathrm{N}_0}$ as possible. Its complement, $1 - \frac{\mathrm{N}_2}{\mathrm{N}_0}$, represents the weight for $\frac{1}{\mathrm{N}_2}$. Finally, to satisfy points number 2 and 3, the following exponent $\frac{\mathrm{N}_0-2}{\mathrm{N}_0}$ needs to be used.

Moreover, the average expected value of the minority parties' shares will be given by:

$$E_{E-\sigma_E} = E - \sigma_E$$

The σ_E, or simply σ, is not a fully independent variable because it is the result of the previously shown inverse relation by Field and Grofman (2007, p. 105). It will be equal to:

$$\sigma = \sqrt{\frac{\frac{1}{\mathrm{N}_2} - \frac{1}{\mathrm{N}_0}}{\mathrm{N}_0}}$$

Therefore, the only independent variables will be N_0 and N_2.

Being σ defined in [0,0.5] in absolute terms, I can impose a further condition, for which $1 \geq \frac{1}{\mathrm{N}_2} \geq \frac{1}{\mathrm{N}_0}$; this derives from the fact that $N_0 \geq N_2 \geq 1$, so, by definition the maximum of σ, could be equal to $\frac{1}{2*\mathrm{N}_2}$, then for $\mathrm{N}_0 = \frac{1+4*\mathrm{N}_2}{\mathrm{N}_2}$:

$$\sigma_{max} = \frac{1}{2 * \mathrm{N}_2}$$

4.3. Synthesizing the party system continuously

At this point, I can start thinking about a most straightforward normal statistical distribution function, in which the ideal share is the most frequent and symmetrical: the higher the frequency and the symmetry, the lower the probability of having smaller or bigger parties in the system, given the same distance from HH. I can now introduce a new hypothesis: being the parties randomly distributed, they could present asymmetries (Doane D.P. Seward L.E., 2011).

Scenarios with a sizeable standard deviation[53] will make the normal distribution function inefficient in describing a given system. Therefore, I want to propose a function with more anchor points[54] such that it fits asymmetric scenarios as best as possible. I define these anchor-points as two new operative terms over HH, where:

a) the ideal share of small parties (ISP) exists as the average of parties having a dimension between 0 and a proxy of HH,

b) the ideal share of the biggest party (IMP) is the average of parties between HH and 1 (100%).

Their frequencies find interesting applications. For example, minority parties and bigger ones have a function of blackmailing and stabilization of the political system respectively; these frequencies can help identify the impact on these ideal points as well as determine the government duration (when considered jointly with the asymmetries of the function that we are defining), contributing to an advancement of relative studies. Other applications could be to more precisely calculate the effect of the block threshold and to solve the open problem of how to predict the average seat allocation in a district[55].

The model can be applied diachronically and synchronically to a party system, allowing comparison and forecasting that can help better understand and identify the asymmetry in the distribution, with unknown applications, as typical of all base researches (Antiseri, 2007). Hence, I can plug three different indicators in the model:

---

[53] For example, a system made of numerous small parties (2-5%) and one with 60%.

[54] *(Taagepera R. , 2008a, p. 35-38, 107-110).*

[55] Discussion done with R.Taagepera on December 11th, 2018.

1) a proxy of HH, ISP, and IMP, equal to the relevant ideal values of the party system, times;

2) the frequency and the probability with which they occur, and then weighting for;

3) their effective incidence on the system.

I need to develop some further tools to get to a formalization of the model. The cumulative function (f_{cum}) of party shares can be derived into a function (f_{der}), representing the probability density that indicates the frequency to have a specific share in each system. The domain of the density function ranges from 0 to 1 (100%); the relative codomain - the y-axis existence - can assume any positive value (being a frequency) such that the summation of all of these values is equal to the probability (0 to 1) to find those shares.

The main innovation in this approach is to work in four dimensions. In the publications following the first work of 1979, Taagepera (2007; 2008a) uses ${N_2}^{-1}$ to express the average percentage or ideal share that "matters" in a given party system, in order to predict the biggest party p_1 and also the others, but only discretely and with a wide range of approximation. My density function will be able to identify how frequent a share is in each system, pushing the boundaries of the existing literature on the argument[56]. Feld and Grofman[57] propose an average value and variance, finding an exact relation between N_2 and N_0 through σ^2, which I will leverage in the model, to find the probability of occurrence - f_{der} - of each party share in the system. These are the four dimensions I am working with.

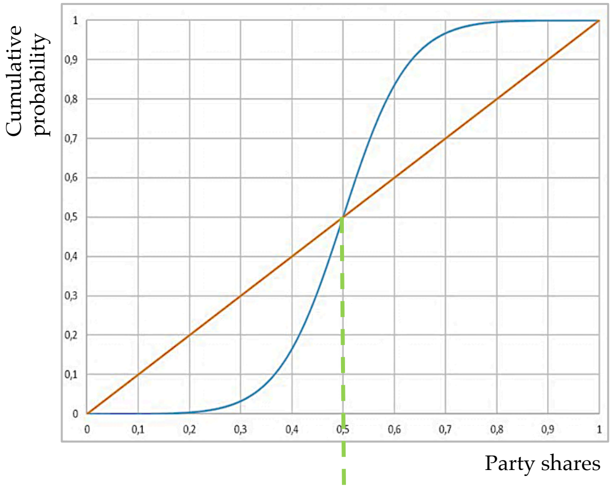
Analytically, if there are no asymmetries in the party system, I can apply Taagepera's [58] law of minority attrition in a new capacity in order to determine the cumulative distribution function f_{cum}.

---

[56] For example, when N assumes value 4, the literature cannot clarify if it indicates four parties with the same share or six parties with shares: 35%, 30%, 15%, 10%, 5%, and 5%.
[57] (2007, p. 105)
[58] (2007b, p. 207-209; 2008a, p. 107-110)

In the graph below I show the graphical feature of f_{der} – the figure 21 - in correspondence of f_{cum} – the figure 20 -. The green dashed line corresponds to HH.

Figure 20 Cumulative distribution function f_{cum} for: $n = 4, N_0 = 2, N_2 = 2$.



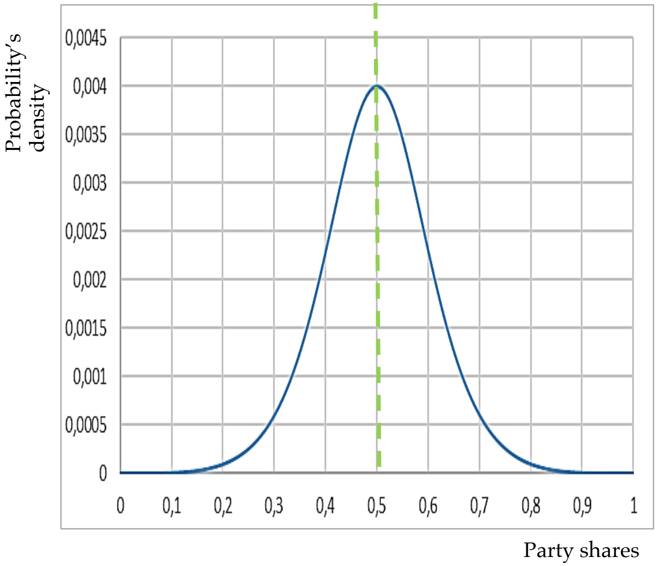


Figure 21 Probability density function f_{der} for parties: $n = 4, N_0 = 2, N_2 = 2$.

According to the theories presented until now, the above function f_{cum} could describe the party system in any political system. The possible shares are on the x-axis, while the sum of the shares empirically predicted are on the y-axis. However, a more articulated f_{cum} , with five inflection points and only three independent variables, can better help visually catch the asymmetries occurring in the real scenarios.

Starting from the proposed generalization of the law of minority attrition, presented in the introduction:

$$\mathrm{s} = \frac{\mathrm{v}^{\mathrm{n}}}{\mathrm{v}^{\mathrm{n}} + (\mathrm{N}_2 - 1)^{1-\mathrm{n}}(1-\mathrm{v})^{\mathrm{n}}}$$

It is possible to get f_{der} by deriving the formula above by v. All derivation passages are in the chapter 4 appendix at the end of the last chapter; the final solution is:

$$f_{der} = \frac{\mathrm{n} * (\mathrm{N}_2 - 1)^{n+1}(-(\mathrm{v}-1)\mathrm{v})^{n-1}}{((\mathrm{N}_2 - 1)^{n}\mathrm{v}^{\mathrm{n}} + (1-\mathrm{v})^{\mathrm{n}}(\mathrm{N}_2 - 1))^2}$$

A crucial property of f_{der} is that $\int_0^1 f_{der} = 1$.

I can then imagine a final probability density function as the sum of three f_{der} built in correspondence of $E_{s_1} = s_1$, another in $E_{E-\sigma_E}$, and the final in E.

$$f_{der\,final} = f_{der}\left(E_{s_1}\right) + f_{der}\left(E\right) + f_{der}\left(E_{E-\sigma_E}\right)$$

The substitution of $f_{der}(i)$ by N_2, is equal to $\frac{1}{E_i}$, as already shown. The bigger the parameter n, the thinner the area (on the x-asis) of f_{der}.

I must introduce the weights w, defined in [0,1], which are equal to the product of E with respectively HH, ISP, and IMP.

Therefore, I must apply the geometric means to obtain each n of f_{der} of the formula above, using the already introduced variables.

The following passages lead to the final function.

$$w_{E-\sigma_E} = \sqrt{E_{E-\sigma_E} * p_{E-\sigma_E}}$$

$$w_E = \sqrt{E * p_E}$$

$$w_{s_1} = \sqrt{E * p_{\sigma_{s_1}}}$$

Hence, it is possible to calculate each n as follows:

$$n_{E-\sigma_E} = \sqrt[3]{\left(\frac{0.25}{\sigma_{E-\sigma_E}}\right)\left(\frac{1}{1-w_{E-\sigma_E}}\right)\left(\frac{0.5}{(0.5-(E-\sigma_E))}\right)^{0.5}}$$

$$n_E = \sqrt[3]{\left(\frac{0.5}{\sigma_E}\right)\left(\frac{1}{1-w_E}\right)\left(\frac{0.5}{|0.5-E|}\right)^{0.5}}$$

$$n_{s_1} = \sqrt[3]{\left(\frac{0.5}{\sigma_{s_1}}\right)\left(\frac{1}{1-w_{s_1}}\right)\left(\frac{1}{(1-s_1)}\right)^{0.5}}$$

I already know that:

$$\sigma_E = s_1 - E \text{ , } (defined\ in\ [0{,}0.5]);$$
$$E_{s_1} = s_1$$

I can express the standard deviation on the smaller and biggest party using the following geometric mean:

$$\sigma_{s_1} = \sigma_{E-\sigma_E} = \sqrt{\frac{0.5-\sigma_E}{2} * \left(0.25 - \frac{0.5-\sigma_E}{2}\right)}$$

The final weights w will be respectively calculated on the previous ones.

$$\sum weights = \sqrt{E_{E-\sigma_E} * p_{E-\sigma_E}} + \sqrt{E * p_E} + \sqrt{E * p_{\sigma_{s_1}}}$$

Following the law of probability of independent events, in which each member has domain [0,1] I can write:

$$\begin{aligned} p_{E-\sigma_E} = {} & \left(1-\frac{1}{\mathrm{N}_2}\right)^2+\left(\frac{\sigma}{\sigma_{max}}\right)^2 \\ & +\left(\frac{1}{e^{(\mathrm{N}_0-2)}}+\left(1-\frac{1}{e^{(\mathrm{N}_0-1)}}\right)-\frac{1}{e^{(\mathrm{N}_0-2)}}\left(1-\frac{1}{e^{(\mathrm{N}_0-1)}}\right)\right)-2 \\ & *\frac{1}{{\mathrm{N}_2}^2}\left(\frac{\sigma}{\sigma_{max}}\right)^2\left(\frac{1}{e^{(\mathrm{N}_0-2)}}+\left(1-\frac{1}{e^{(\mathrm{N}_0-1)}}\right)\right. \\ & \left.-\frac{1}{e^{(\mathrm{N}_0-2)}}\left(1-\frac{1}{e^{(\mathrm{N}_0-1)}}\right)\right) \end{aligned}$$

$$p_E =$$

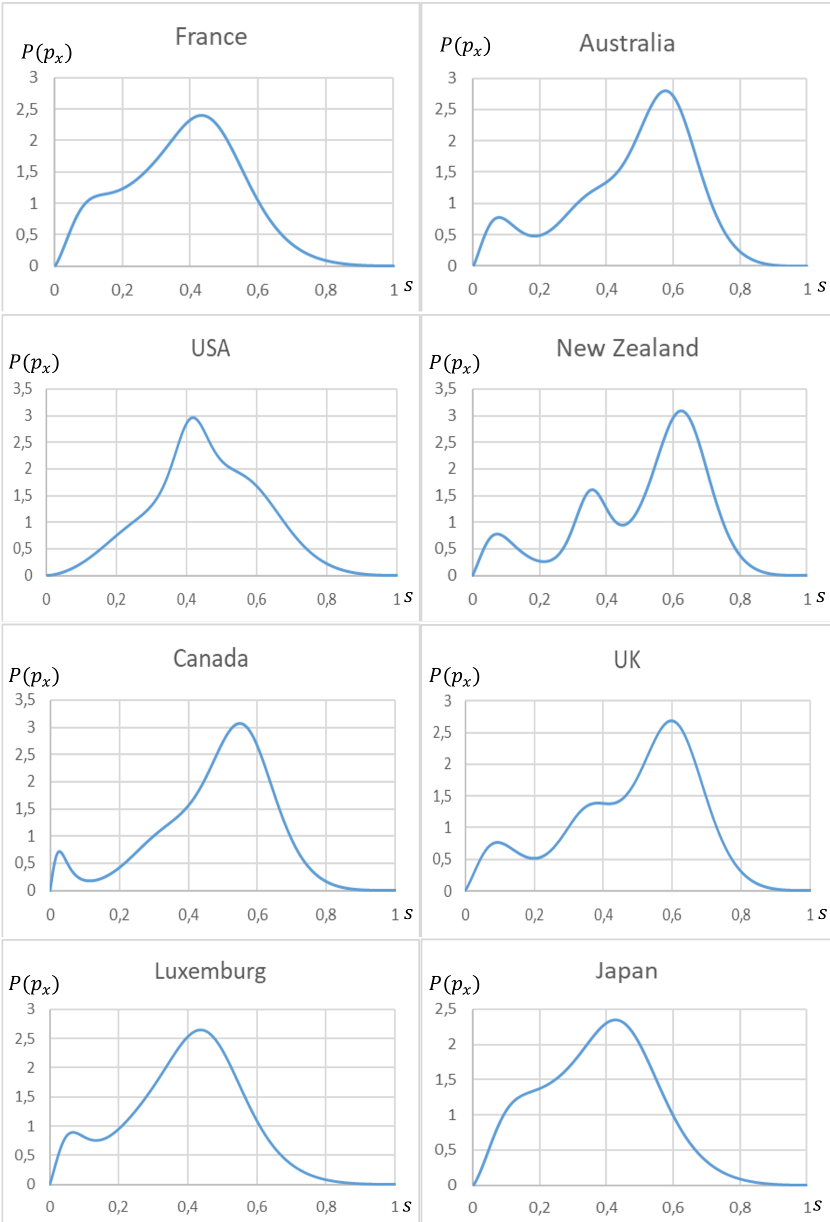
$$p_{\sigma_{s_1}}=\frac{1}{{\mathrm{N}_2}^2}+\left(\frac{\sigma}{\sigma_{max}}\right)^2+\frac{1}{e^{(\mathrm{N}_0-1)(\mathrm{N}_0-2)}}-2*\frac{1}{{\mathrm{N}_2}^2}\left(\frac{\sigma}{\sigma_{max}}\right)^2\frac{1}{e^{(\mathrm{N}_0-1)(\mathrm{N}_0-2)}}$$

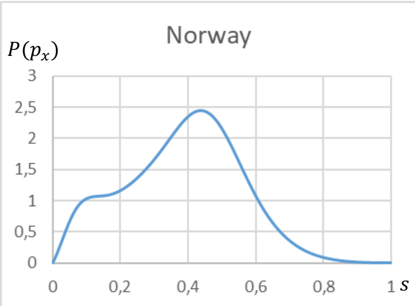
Below in Table 9 are shown the applications of this statistical function for each of the 16 countries which are in common across the following sources (Taagepera R. , Parsimonious Model for Predicting Mean Cabinet Duration On the Basis of Electoral System, 2010, p. 270); (Taagepera R. , Predicting Party Sizes, 2007b, p. 288, 291), as below reported:

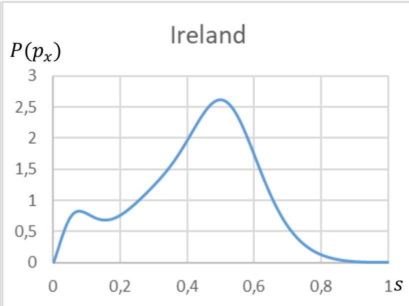
Table 9 Noteworthy political variables for 16 countries. Sources: (Taagepera R. , 2010, p. 270), (2007b, p. 288, 291); and my elaborations.

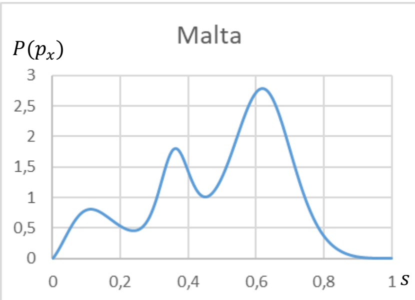
country	N_0	N_2	*Actual* s_1	*Skewness [0,1]*
FRANCE (1958-2002)	6.7	3.43	0.44	0.823
AUSTRALIA (1919-1996)	3.7	2.22	0.51	0.812
NEW ZELAND (1890-1996)	3.5	1.96	0.57	0.815
CANADA (1878-1993)	4.4	2.37	0.56	0.878
USA (1878-1993)	2.5	2.40	0.62	0.637
UK (1922-1997)	6.4	2.11	0.53	0.792
LUXEMBOURG (1919-1999)	5.5	3.36	0.41	0.840
JAPAN (1928-1996)	10	3.71	0.54	0.829
NORWAY (1945-1997)	6.3	3.35	0.47	0.823
IRELAND (1922-1997)	8.2	2.84	0.48	0.828
PORTUGAL (1975-2002)	6.9	3.33	0.43	0.821
MALTA (1947-1987)	3.2	1.99	0.53	0.776
SPAIN (1977-2004)	12.8	2.76	0.5	0.801
FINLAND (1907-2003)	6.8	5.03	0.27	0.869
SWEDEN (1948-1994)	5.7	3.33	0.47	0.830
ITALY (1895-1994)	5.1	4.91	0.41	0.877

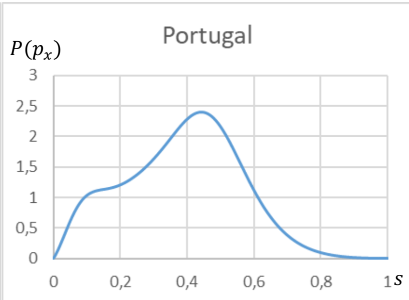
Figure 22 Probability density functions of 16 countries' party systems.

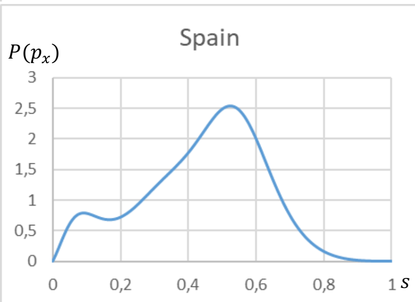


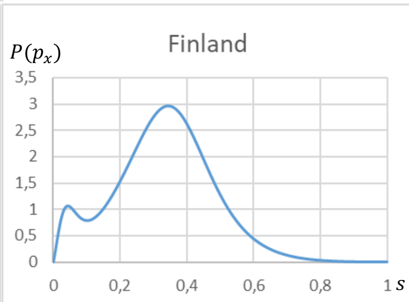

Norway
P(px)
3
2,5
2
1,5
1
0,5
0
0
0,2
0,4
0,6
0,8
1
s

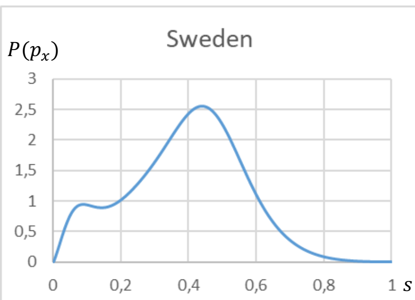

Ireland
P(px)
3
2,5
2
1,5
1
0,5
0
0
0,2
0,4
0,6
0,8
1
s


Malta
P(px)
3
2,5
2
1,5
1
0,5
0
0
0,2
0,4
0,6
0,8
1
s


Portugal
P(px)
3
2,5
2
1,5
1
0,5
0
0
0,2
0,4
0,6
0,8
1
s


Spain
P(px)
3
2,5
2
1,5
1
0,5
0
0
0,2
0,4
0,6
0,8
1
s


Finland
P(px)
3,5
3
2,5
2
1,5
1
0,5
0
0
0,2
0,4
0,6
0,8
1
s


Sweden
P(px)
3
2,5
2
1,5
1
0,5
0
0
0,2
0,4
0,6
0,8
1
s

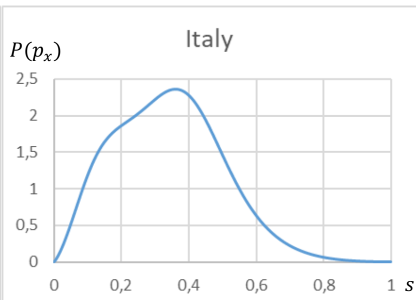

Italy
P(px)
2,5
2
1,5
1
0,5
0
0
0,2
0,4
0,6
0,8
1
s

Figure 22 shows the probability density functions of 16 countries. The x-axis represents party shares, by definition existing from 0 to 1, and the y-axis represents the relative frequencies, always greater than or equal to 0.

At a first glance, these graphs show the relative frequencies by each party share, in a comprehensive way, synthesizing a given party system. Considering the proportions between the y-axis values, I can visually estimate what are the proportions between the product of the number of parties and any x-axis party share, occurring in a specific country's party system. What does comprehensive mean? This means that the relative frequency shown represents the summation of the occurrence of all parties having the same party share in common.

Being a probability density function, there is not an instant way to count exactly the number of parties by party share, since the infinitesimal frequency by party share is measured instead. This is in fact useful to calculate a much stable and precise measure of any range of party shares, and not of a single or specified party share for example for a single election.

Why not a discrete measure? Suppose to analyze the US distribution (1878-1993). Table 9 reports s_1 as equal to 0.62, although in figure 22 there is a pick around the point with abscissa 0.42. Looking with more attention to the graph, it is possible to recognize a homogeneous bending[59] of the curve in the range of points with abscissa from around 0.5 to 0.7. This means that even though a value with abscissa 0.62 exists on average, there is an asymmetry (skewness) such that the mode (but also the median value) does not overlap with the average. This happens in almost all cases.

In case of a simple histogram, with absolute frequencies on the y-axis, and the party shares on the x-axis, it would be simply impossible to depict the average "ideal-typical" party shares in an interval of time. For example, it would be impossible to represent in a single histogram, with summation of frequencies equal to 100, the summary of results of two consecutive elections for the following two-party systems: A: 40%, 30%, 20%, 10%; and B: 40%, 30%, 30%; in particular, some challenges would be: how to allocate a party with a 10% or a 20% share in scenario A in an "average" with the first or second party with a 30% share in scenario B? What kind of average to apply? Which procedure to use in order to organize these averages?

[59] having a similar second derivative.

A party system in a certain election could be indeed represented using a histogram - as before settled - however, determining how party shares are distributed on average would be meaningless for whatever probabilistic calculus, because without a probability density function it is by definition impossible to extend a specific scenario to the other party shares because it is just as an instantaneous picture. For example, with reference to previous scenario A, if I wanted to evaluate the probability of a block threshold impacting on the party system, I would need to know the probability that such party system would produce a share from 0 to 10%, requiring use of this new probability density function, as nothing would be said by a histogram representing a specific scenario of that system.

The last step is to determine how to practically calculate the probability that parties exist and how many within a certain x-axis interval. This will be simply done calculating the defined integral of the new probability function in that interval, and then dividing this integral value by the integral from 0 to 1 of the same new probability density function. Furthermore, in order to calculate the number of parties existing for this party share interval, I divide this last result by the average party share in the same interval.

In addition to estimating the impact of block threshold as mentioned above, another application of the five inflection point model could be to evaluate the general asymmetry among parties' shares, derived from this function, thus allowing the calculus implementation of the cabinet's life, using such asymmetry as exponential corrective to the effective number of parties in Taagepera's cabinet life formula. Some further considerations are drawn in chapter ten.

Chapter 5

Using the Eulerian Beta function for the Downsian model

5.1 Introduction

This chapter aims to answer the question: how is it possible to better understand - quantitatively - the left-right parties' collocations on the ideological continuum? The answer to this question is crucial to understand if there are some uncovered ideological spaces that could represent a potential source of votes increase. It can also help identify the possibility of present and future alliances among parties for election purposes, as well as to improve cabinet formation and duration models. It could also be used to evaluate the game theory maximization of the electoral system, not only in function of political and institutional terms but also in terms of ideological cleavage.

This chapter introduces the Eulerian Beta curve to illustrate the ideological party's positioning, as never done before. Therefore, it can be considered a "buffer" chapter, establishing one of the propaedeutic indicators of the following two chapters.

My aim is to find a simplified unimodal function that can identify the ideological influence areas for each party. It is not possible to use the non-monotonic function (multi-picks) shown in the previous chapter, since it is not unimodal: it would be misleading to have multiple picks of the ideological influence areas for each party without understanding its "core business".

Here I present a revised model to quantify and simplify the ideological function describing the left-right space occupied by any party. The unparalleled operative tools used are beta functions, which will improve the most recent approaches (Adams - Merrill III - Grofman J. -S.-B., 2001, p. 15-51). This will simplify the calculus around the interaction of the ideological positioning with the electoral mechanics, as shown in the following chapters.

In order to identify the party positioning, I have sourced the values of electors' self-positioning on the left-right continuum from the post-electoral Italian database ITANES (1948 - 2013) from 1994-2013. Firstly, I register that in the 2013's election, more than 80% of electors located themselves on the left-right continuum, confirming that the stability of the ideological identification through time is constant at 98.8%. Secondly, I apply the Beta probability density function to the self-positioning values, grouping these by party. This results in a unique beta function by party (by election and chamber) through the parameters alpha and beta characterizing the Eulerian Beta function.

5.2 The theoretical model

As a first step, I want to check if there is any electors' self-positioning on the left-right continuum; if so, and in absence of any trends of large fluctuation, then I can consider the ideological positioning of the parties meaningful together with the positional party competition and the connection with electoral flows, respectively shown in the next two chapters.

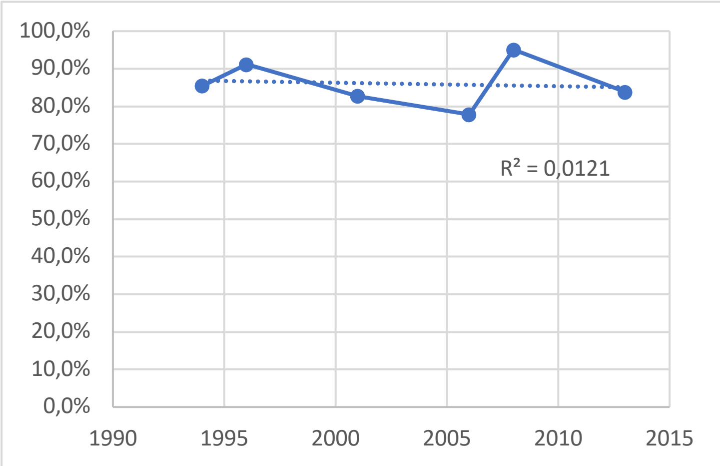
I then present the percentages of respondents who declared a placement in the left-right axis sourced from the surveys ITANES (1948 - 2013) in the period 1994 to 2013. As shown in the figure 23 below, these percentages have fair stability over time – equal to the complement of the R^2 of this time series at 98.8% - other supporting elements are the significance of the constant, which is at least 99.99%, and the insignificance of the temporal independent variable, which results 83.57% (being the complement to the probability given by the t value equal to -0.2212), as expected.

In addition, the self-positioning of almost always more than 80% of electors on the left-right continuum confirms the classical theory of elector positioning from Downs To Grofman (Downs, 1957) (Grofman, 2004) (Adams - Merrill III - Grofman J. -S.-B., 2001); nevertheless, some[60] critic this dimension is still valid and constant over time.

Furthermore, a more stable ideological identification of electors than the typical party behavior[61], which tends to be more independent (Inglehart - Klingemann, 1976), will be helpful in the analysis in the last part of chapter seven, related to the electoral flows' impact. *1*

[60] *(Giddens, 1994);(Bobbio, 1994).*
[61] *(De Sio, 2011, p. 83).*

Figure 23 Percentage *of electors which position themselves on left-right dimension in the Italian elections 1994-2013.*



Hence, I can proceed to the graphical bi-dimensional representation of the party positioning on left-right x-axis, formalizing what was firstly proposed by Downs (1957) and followed by the foundational trick of approximation used by Merril III, Adams, and Grofman (2001, p. 15-51)[62], overcoming a limit of this most recent tool which unfortunately misses the anchor points (0,0) and (1,0)[63] which the parties' positional curve should cross.

Like done by the latter authors, I am representing these curves in a simplified unimodal form, unparalleled embodied by the probability density function Beta, such that $\mathrm{P}(\mathrm{Beta})_i = \frac{v^{\alpha-1}(1-v)^{\beta-1}}{\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\,dv}$ existing in [0,1].

By definition, the expected value $E = \frac{\alpha}{\alpha+\beta}$ and the mode in $m = \frac{\alpha+1}{\alpha+\beta+1}$. This Beta function allows to cast into system the last two equations by substituting the values E and m for each party positioning obtained from the elector ideological self-positioning grouped by party.

---

[62] Mainly concerning an equilibrium model of party competition.

[63] Their proposed function is $U_i(k) = -a(x_i - s_k)^2 + bt_{ik}$, i is the i-th party, a and b are the parameters, s_k is the party-candidate position, x_i is the voter position, and t is the non-positional political issues. *Ibid* (2001, p. 17,22,31)

Since the denominator of $\mathrm{P}(\text{beta})_i$ is equal to $d_i = \int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\,dv$, it will be necessary to multiply each function $\mathrm{P}(\text{beta})_i$ for a coefficient $r_i = \frac{p_i * \sum_1^{N_0+not\ vote} d_i}{d_i}$ such that the area subtended by the beta function of each party would be equal to the percentage of votes obtained by the same party (the non-vote can also be considered as a party), obtaining $\mathrm{PE}(\text{beta})_i = \mathrm{P}(\text{beta})_i * r_i$. This results in a unique beta function by party (by election and chamber).

5.3 The applications

All the probability density distribution functions for all parties of chamber and senate from 1992 until 2018 are represented below, from figure 24 to 39.

These have been built from the relative seats' allocation proportional to the ordinates (y-axis) and the ITANES data introduced before. The integral of each curve below has been calculated on the basis of the seats obtained by each party respectively, in the chamber or the senate, alternatively to the representative organism considered in the graph, determining a weighting by seats.

In conclusion, with these curves it will be possible to identify the hegemony for each party on the left-right continuum for each scenario: the party that achieves the highest ordinate, for a given abscissa of left-right ideological location, will represent the leader for that particular ideological position.

Figure 24 Chamber positioning 1992 Sx(0)-Dx(1) (weighted by seats).

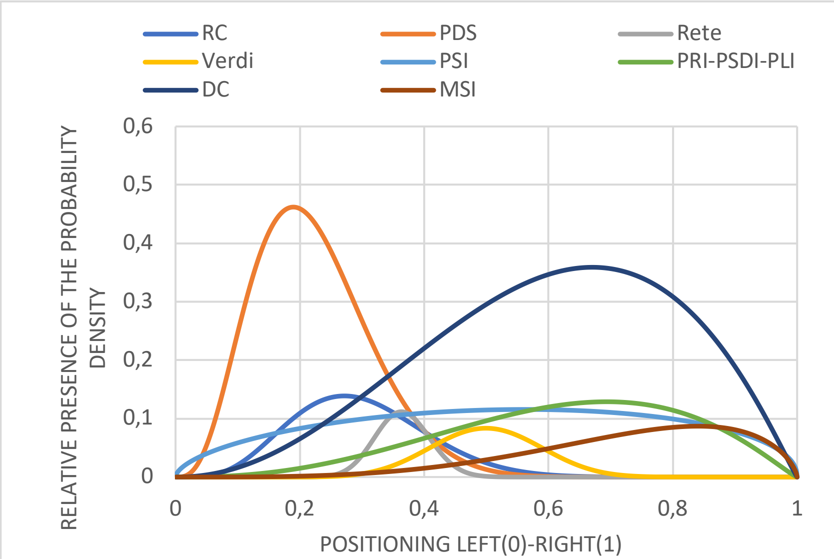


Figure 25 Senate positioning 1992 Sx(0)-Dx(1) (weighted by seats).

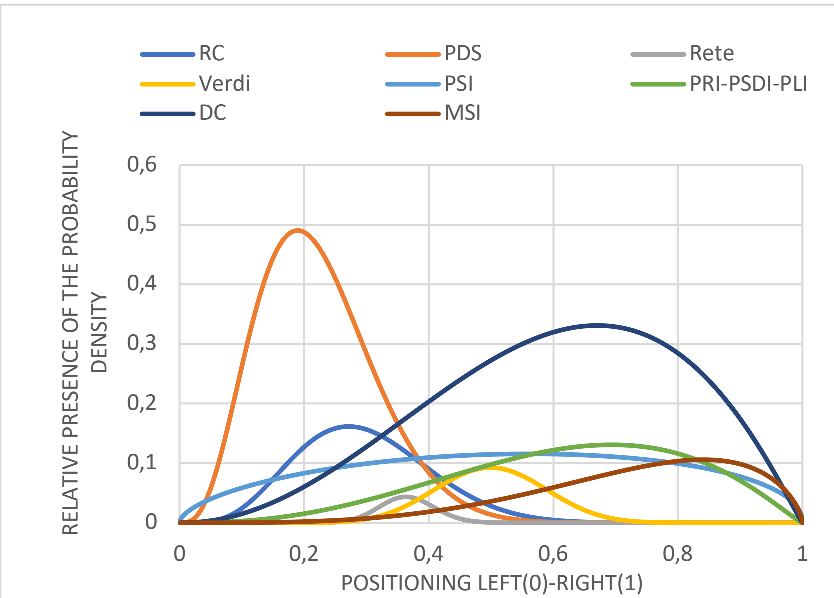


Figure 26 Chamber positioning 1994 Sx(0)-Dx(1) (weighted by seats).

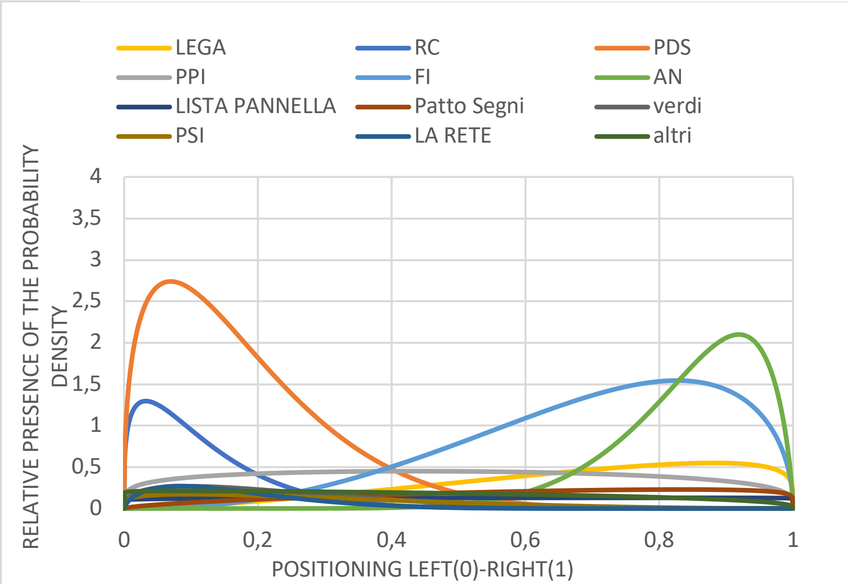


Figure 27 Senate positioning 1994 Sx(0)-Dx(1) (weighted by seats).

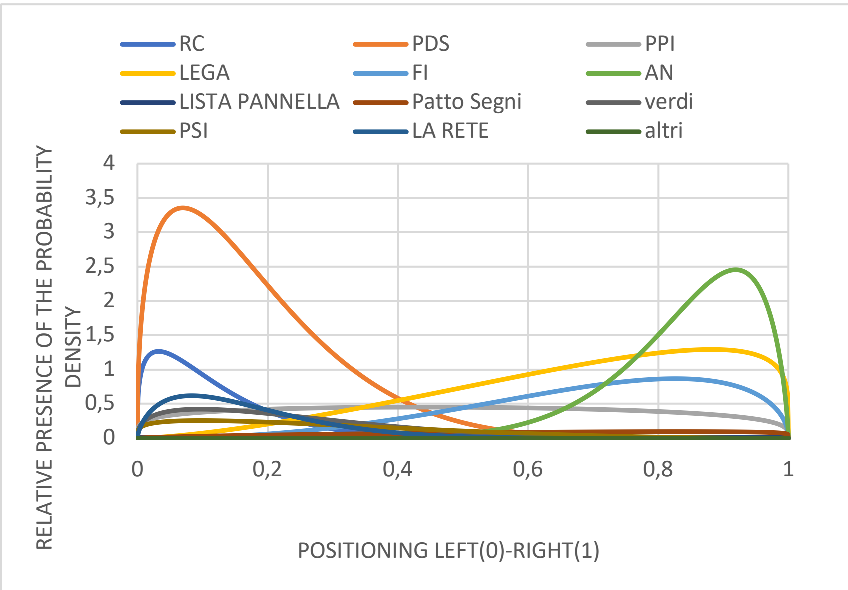


Figure 28 Chamber positioning 1996 Sx(0)-Dx(1) (weighted by seats).

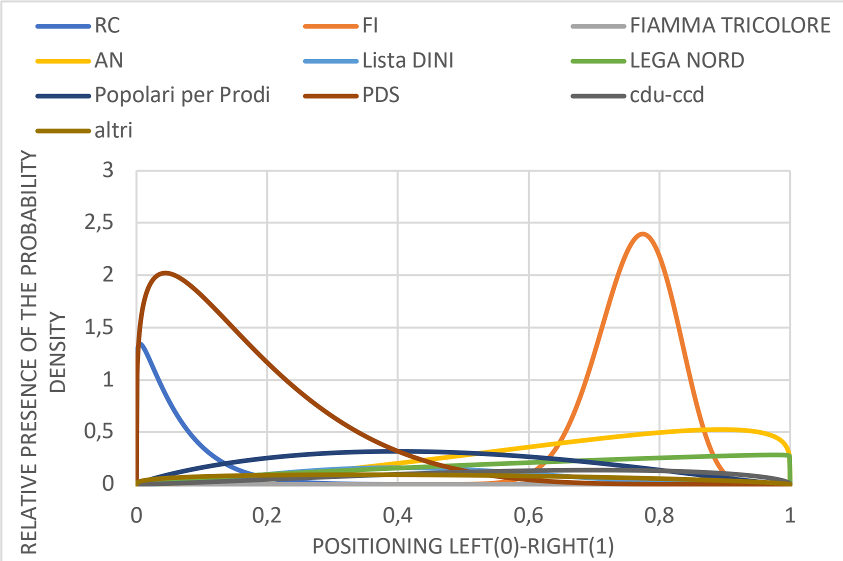


Figure 29 Senate positioning 1996 Sx(0)-Dx(1) (weighted by seats).

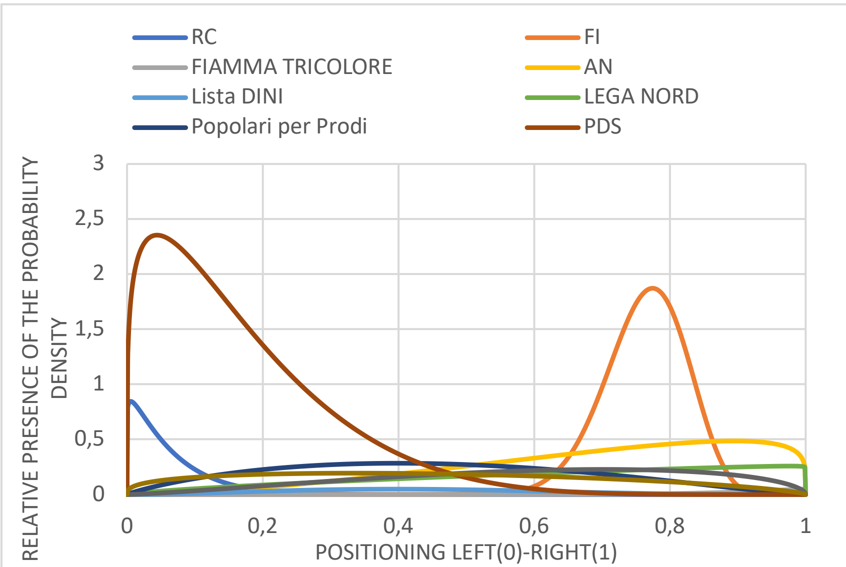


Figure 30 Chamber positioning 2001 Sx(0)-Dx(1) (weighted by seats).

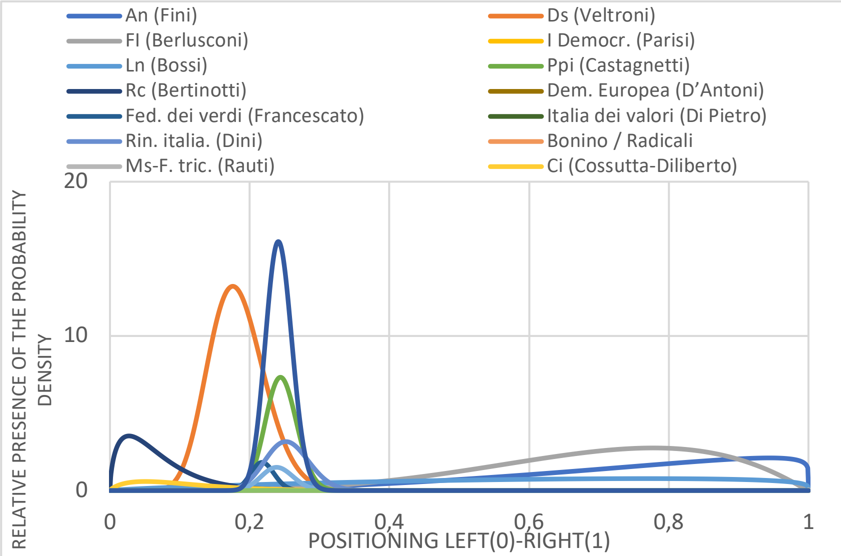


Figure 31 Senate positioning 2001 Sx(0)-Dx(1) (weighted by seats).

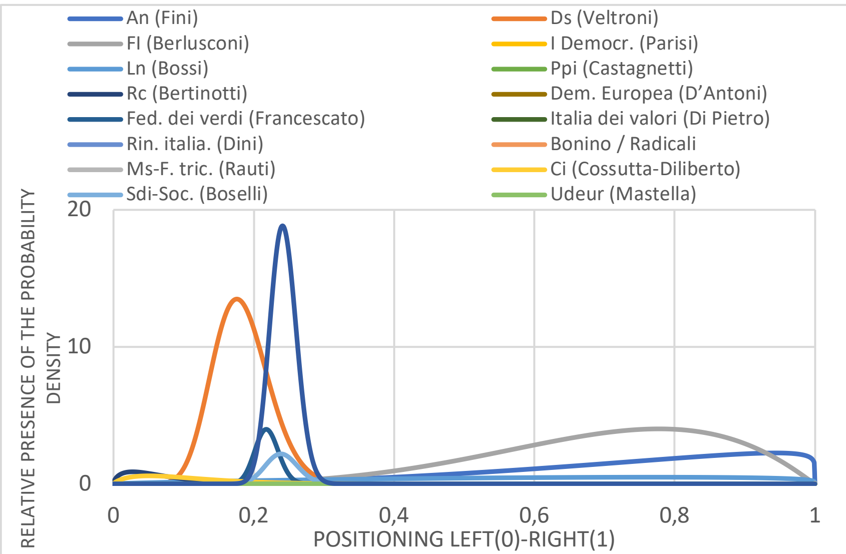


Figure 32 Chamber positioning 2006 Sx(0)-Dx(1) (weighted by seats).

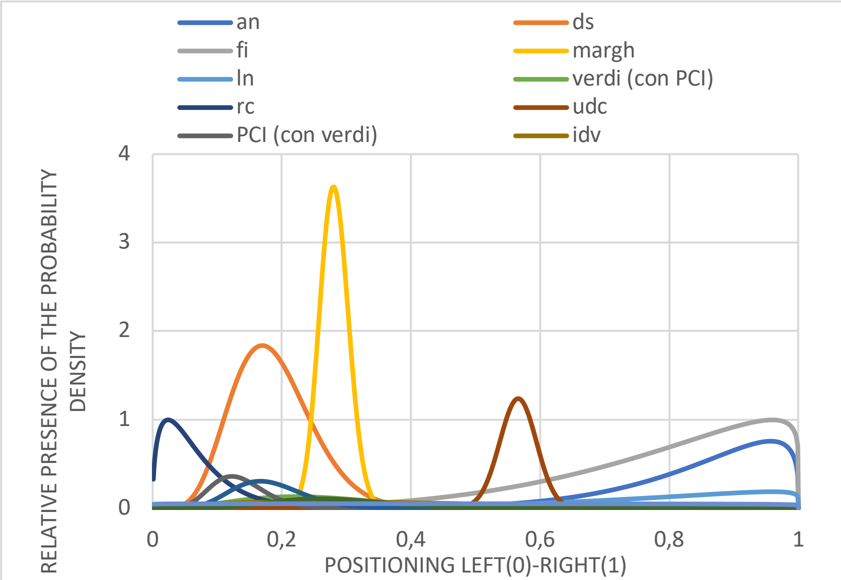


Figure 33 Senate positioning 2006 Sx(0)-Dx(1) (weighted by seats).

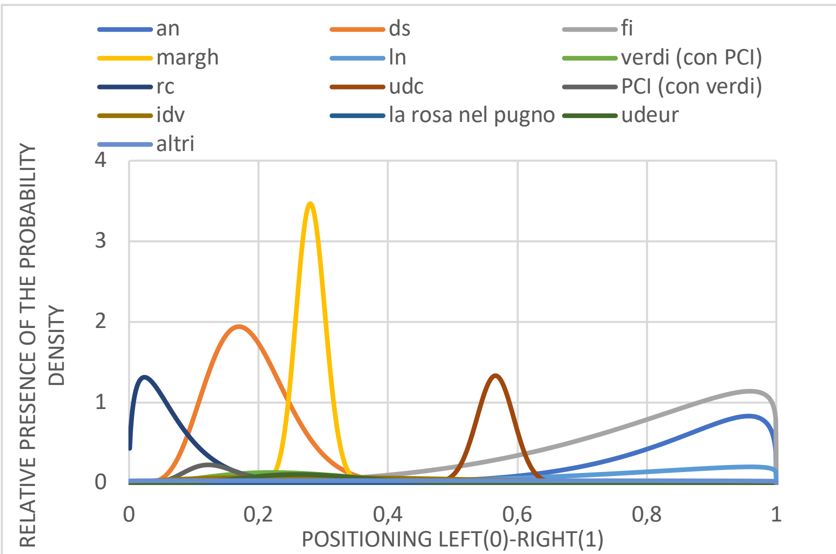


Figure 34 Chamber positioning 2008 Sx(0)-Dx(1) (weighted by seats).

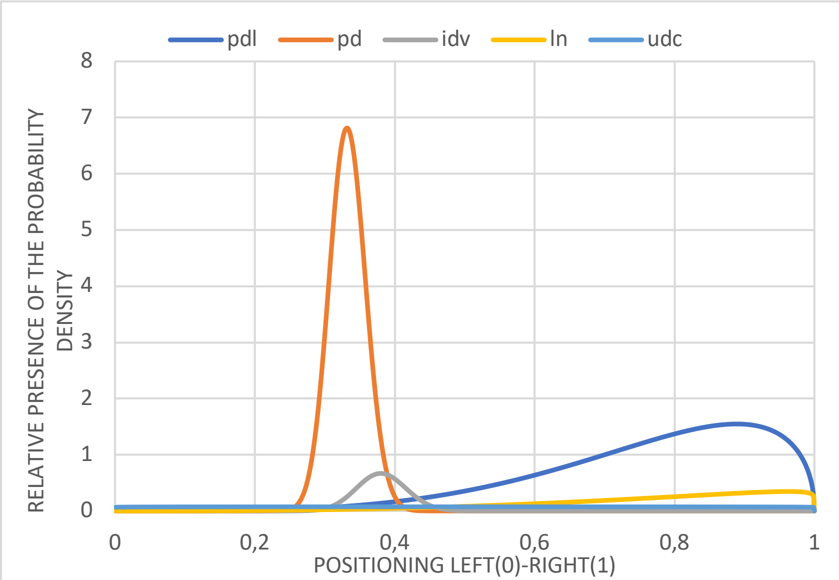


Figure 35 Senate positioning 2008 Sx(0)-Dx(1) (weighted by seats).

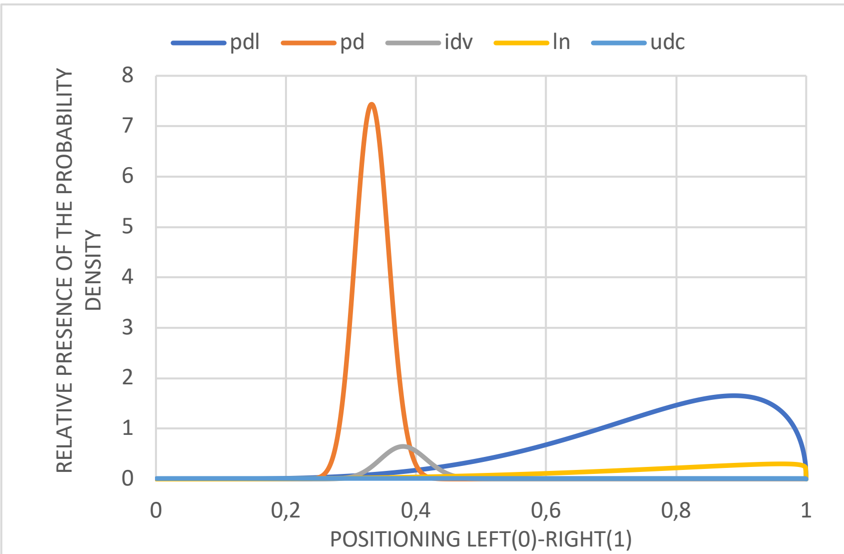


Figure 36 Chamber positioning 2013 Sx(0)-Dx(1) (weighted by seats).

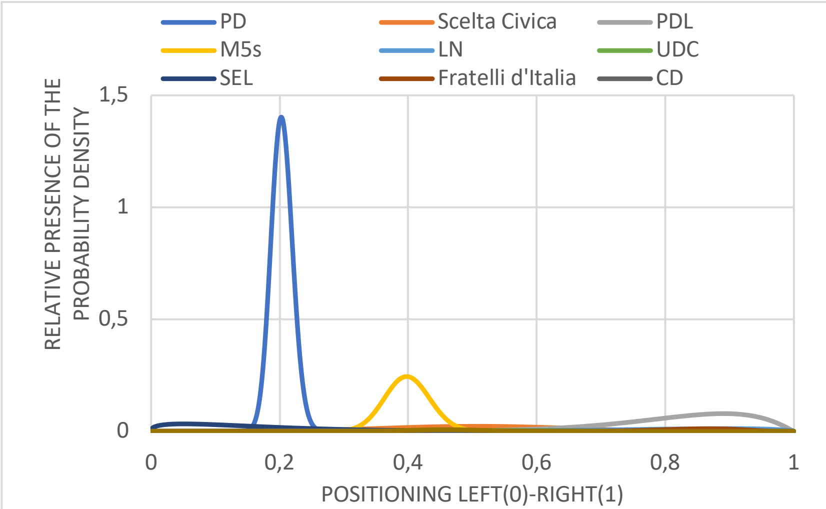


Figure 37 Senate positioning 2013 Sx(0)-Dx(1) (weighted by seats).

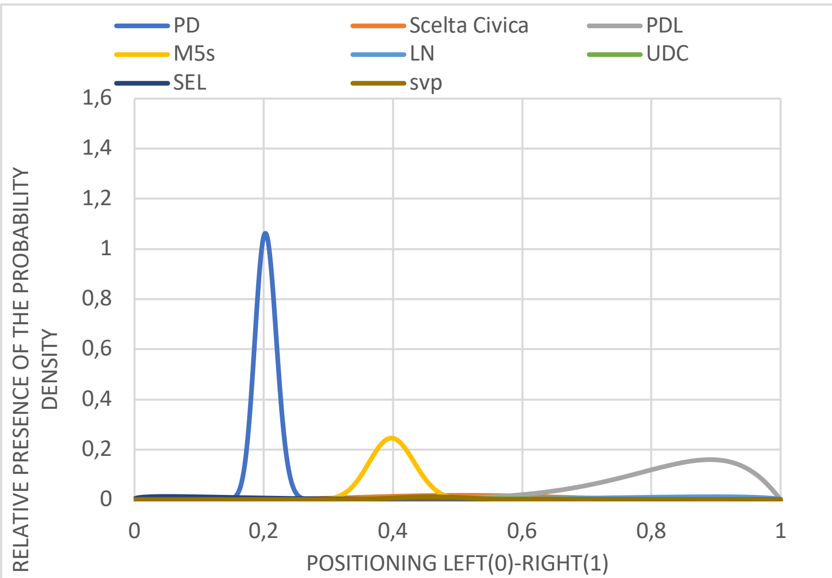


Figure 38 Chamber positioning 2018 Sx(0)-Dx(1) (weighted by seats).

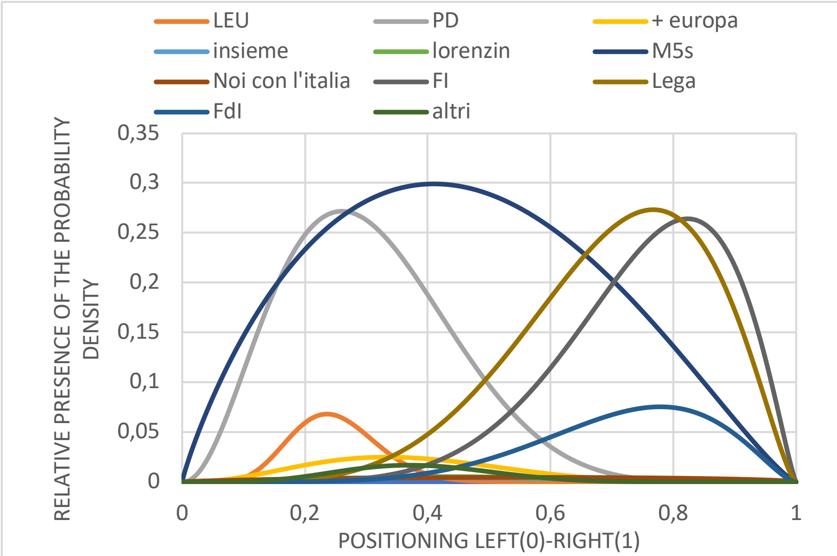
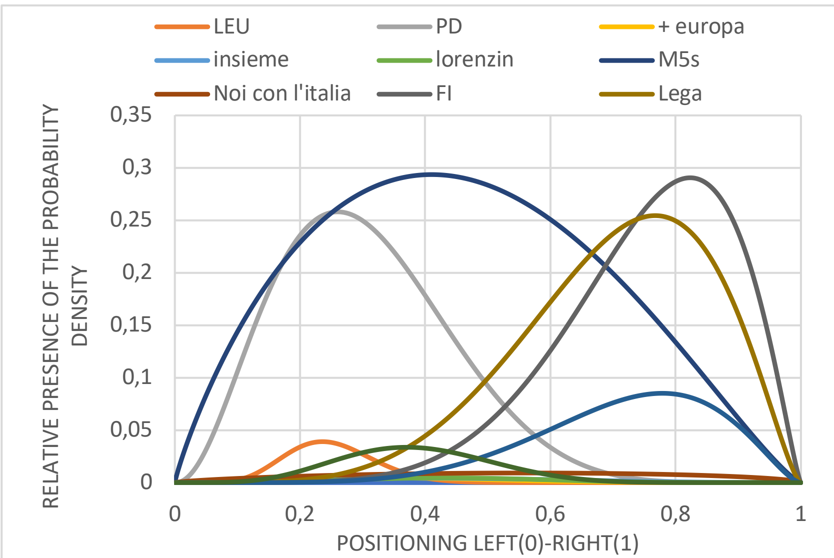


Figure 39 Senate positioning 2018 Sx(0)-Dx(1) (weighted by seats).



The graphs above are fairly self-explicative. For exemplificative purposes, I will comment on the last elections illustrated, which are the ones in 2018.

The variance of the positional curves of the PD, M5S and Lega parties has increased compared to 2013, implying an increase of their respective permeability; instead, FI has maintained roughly the same one. Moreover, in 2018 M5S’ increase in variance has covered almost all of PD’s influence's area (left-right), which can be mainly explained by the raise of the imperative-valence (non-positional) issues which have involved the party system as well as PD's electors.

Nevertheless, for the same elections, the PD has widened its influence's area on the left-right continuum, however without implying an increase of votes, mostly because of the use of positional issues more on the right, rather than the use of valence issues, as done by M5S; in addition, it should be considered that leadership credibility could also perish.

Chapter 6

Downsian competition, number of parties, and the disproportionality index

6.1.Introduction

This chapter aims at integrating and qualitatively correlating solid and essential political sciences theories exploring and deepening the relation between electoral systems and party competition, which is still more than a theoretical puzzle.

This puzzle is based on the theory that the more majoritarian the electoral system, the lower the number of parties, the more centripetal the party positional competition[64]. I find it of enormous interest to investigate the correlation and the impact that an electoral system could have on the positional competition in pushing a change in the number of parties, depending on whether it is more proportional or majoritarian (respectively given by a lower or higher index n, which represents the disproportionality (Taagepera R. , 2007b, p. 204-207)). Furthermore, I endeavor to measure how a change of the Effective number of parties (N) could impact the positional competition and calculate the different dis-representation effect given by the same electoral system, and finally how a change of one of these variables impact the other two.

In detail, I proceed to quantify the reflection made by Bernard Grofman in the article "Downs and two-Party convergence" (2004), in which the Downsian theory (Downs, 1957) of parties' ideological convergence has been re-read in function of how tight the initial assumptions are with regards to the concrete political and institutional scenarios. This chapter blends existing studies on: 1) how the Downsian convergence would vary by electoral system (Grofman, 2004, p. 26, 31), 2) the number of parties competing in an election (Id. (p. 26-8)), 3) the positional, non-positional, and majoritarian competition (De Sio, 2011).

I apply this innovative blended new model to the system of party competition in a logical-qualitative approach, as never done before. The resulting model will be more robust thanks to the fact that the

[64] The non-positional competition (Lewin, 1935); (Stokes, 1963); (De Sio, 2011, p. 48) and majoritarian competition (Robertson, 1976); (Budge - Farlie, 1983); (Budge - Robertson - Hearl, 1987); (De Sio, 2011, p. 57-8) will also be taken into consideration. The chapter will scrutinize these elements.

institutional variables are linked to the parties and positional ones: modifying one will change all the others.

The aim is to use this model in comparative statics – as introduced before – where the final equilibrium is the product of the interlocked multivariate system of causality, rather than of the cleavage between dependent and independent variables. This allows to: 1) measure how a specific electoral system can modify the party ideological positioning on the left-right continuum, 2) know how many parties would be in the political space, and their location, with no disproportionality due to electoral systems, 3) quantify the countervailing effect of the electoral system in charge, introduced only qualitatively by Sartori (2003, p. 61-2). As introduced before, in presence of an "*Ideal-typical"*[65] plurality or proportional system[66], as discussed by existing theories including those on party competition, my theory can overcome the limits of the pure FPTP and proportional electoral systems, as it is able to also catch the shades amongst them. A pillar theory underlying my model is that the more proportional an electoral system, the more the parties tend to a centripetal competition (Sartori, 2003, p. 60-3), implying that minor parties tend to assume extreme ideological positions to be more visible.

This electoral dynamics reasoning can also be transferred into ideological terms, since the growth-survival of some parties find a fertile environment in a purely proportional electoral system (as there are fewer barriers to entry) therefore parties should logically occupy all the ideological vacuums available, probabilistically smoothly spreading on the left-right positional continuum; conversely, a robust majoritarian electoral system will tend to produce a bipolar competition. The pillars of political science supporting these mechanics are: the laws of Duverger (1951; 1954, pp. pp. 247, 269; 1955, p. p. 113), the party competition of Downs (1957), and relative upgrades offered by Rae (1971, p. 95), Riker (1982, p. 760) and Sartori (2003).

The measure of disproportionality used in this chapter will be n_2 as introduced and defined in the chapter two, nevertheless in absence of any changes of the simple n, can be simply used the latter.

To verify the relation between n_2 and party competition dynamics, I have collated the data from the in-depth qualitative analysis in the period

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[65] referring to the Weberian concept of the ideal-types (Weber, 1972).

[66] as well explained by Sartori with regards to electoral mechanics in "pure" plurality systems (2003, p. 57-64), the degree of disproportionality n (Taagepera R. , 2007b, p. 204-207) is quantitatively different, even though formally all districts present the same electoral system with a single winner.

analyzed for Italy's seats from votes (from 1992 to 2018) and the positional party location from the previous chapter in the same period. In order to reproduce the party positioning of the years 1992 and 2018, I also used electoral flows, creating a dataset, using as primary sources the ITANES archive (1948 - 2013) and others (Diamanti - Mannheimer, 1994, p. 114), (D'Alimonte R. - Chiaramonte A., 2010) (Bartolini - Chiaramonte - D'alimonte, 2002) (Carrieri, 2018) (De Sio - Paparo, 2014) (SWG, 2018) (IPSOS, 2018) (De Sio - Paparo, 2014).

This chapter can only provide an in-depth qualitative and directional analysis, due to the complex calculus procedures and the heavy load of coherent information needed, which would have come from different international databases – currently not available. Hence, this chapter uses a logical qualitative approach, but not a quantitative nor a predictive one. In particular, the final qualitative models suggest a tri-dimensional relation between n, the average weighted positional distance px, and N; and a correlation between px and the Effective number of parties (weighted on the electoral system). A certain change of $\boldsymbol{n_2}$ would have an impact: mainly determined by positional party competition (conditional to the non-positional one), secondly by the degree of bi-polarization of the party system, and lastly by the weighted ideological distance px.

6.2.The Puzzle

I propose an integrated analysis, which puts to system the different pieces of knowledge from political science theories and more generally in social sciences, running through: the social choice (Curini, 2015) (Osborne, 1995-2000), electoral systems (Colomer J. , 2004; 2005) (Taagepera R. , 2007b) and the theories of the parties competition (Adams - Merrill III - Grofman J. S., 2005) (Merrill III - Adams, 2001) (De Sio, 2011), particularly stressing and starting on the theoretical interactions that occur among electoral systems and party format as a milestone placed by Sartori (1987; 2003, p. 43-68), readapting these in function of a refined model of permeability (De Sio, April 2006; February 2008; 2011, p. 82-107,111-127).

The research question is how we can recognize some relevant variables of party competition such that it would be influenced by the index of disproportionality n_2 seen before. Taking some recent literature, and suggesting some upgrades, I use a party competition model based on three dimensions[67]: 1) positional, 2) non-positional, and 3) majoritarian. This model could be complementary to that of permeability produced by De Sio (2011), as for this specific research question the permeability is applied to each party and not aggregately; moreover, I propose an endogenic-simplified version of permeability, obtained directly from the probability density function of each party[68]. This approach presents the advantage of catching the party influence area and infinitesimal left-right position, interaction, and the increase/reduction over time of non-positional issues for each party.

I explore into more details each of the three dimensions listed above.

1) Positional competition was firstly introduced in the economic competition theory discussing the linear distance between companies, which would have implied the convergence towards the same point in this one-dimensional territorial space, as per Hotelling (1929). The same concept was used by Downs (1957) to identify a left-right continuum (bi-dimensional) arena where all electors would have an agreement on the positioning of political parties. As demonstrated in the previous chapter, I deem this concept valid, together with Downs' theory of a convergence towards the centre between two parties in a majoritarian competition, as just introduced by Hotelling. This concept was criticized by Arrow (1951) and Grofman (2004), who set a series of conditions necessary for this to happen, such as empirical critiques observed on American elections at a local and national level. One for all, see Budge et al. (2001). See De Sio (2011, p. 18-39) for a larger disquisition about party positioning and political equilibrium.

2) Non-positional competition has been introduced by Stokes (1963)[69], borrowing the term of «valence issues» - firstly used by Kurt Lewin (1935) – in relation to «those that merely involve the linking of the parties with some condition that is positively or negatively valued by the electorate» (Stokes, 1963, p. 373). Some issues exist on which

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[67] In agreement with De Sio (2011) and the recent literature presented in this chapter.

[68] In the complete model, De Sio puts the permeability as a function of political involvement (De Sio, 2011, p. 12,62-81,96-107), defined synthetically as the «interest for politics» and the «knowing of politics» (p. Ib., 144), in which $v = Z + Hi$ (p. Ib., 122-3), being v the political permeability, Z a constant and H the parameter relative to the political involvement i. The parameters Z and H were estimated using data from 20 elections among the USA, France, and Italy (Ib., 128-231).

[69] And further developed by Stokes (1992) and Clarke et al. (2004) for an autonomous approach.

all electors agree to be satisfied and then the question switches from what to do (about alternative policies) to who is the most capable to fulfil those requests (De Sio, 2011, p. 48)[70].

3) The term Majoritarian competition[71] is used to indicate a blurry continuum that exists in between the previous two types of competition. All intermediate states of competition – issues - allow a majoritarian consensus to exist on these issues, even though not unanimously, such that: 1) probably these are used to the advantage of minority parties on the ideological side; 2) the parties compete to own certain issues.[72] This happens because of the *issue priority*, that sees parties giving priority to themes which they know have a good reputation (Robertson, 1976) (Budge - Farlie, 1983) (Budge - Robertson - Hearl, 1987). «Therefore, a particular theme becomes naturally attributed by the electors to a certain party, which determines a true possession (*issue ownership*) » (De Sio, 2011, p. 57).

Considering the results of the district elections (Ministero dell'Interno, 2018) I will be able to quantify the degree of structuration of the party system on the basis of parties' concentration over the territory - on a district basis - (Sartori, 1987, p. 58-61). The hypothesis is that the more concentrated the parties are, the least impact on the party numbers and then on spatial competition by a disproportional system (Rae, 1971, p. 95) (Riker, 1982, p. 760); as Sartori says:

> «Rule 3. Conversely, a two-party format is impossible-under whatever electoral system-if racial, linguistic, ideologically alienated, single-issue, or otherwise incoercible minorities (which cannot be represented by two major mass parties) are concentrated in above-plurality proportions in particular constituencies or geographical pockets. If so, the effect of a plurality system will only be reductive vis-a-vis the third parties which do not represent incoercible minorities.» (Sartori, 2003, p. 59)

To understand the problem of positional competition, I take a

[70] Particularly Clarke et al. (2004) proceed to identifying three types of political valences: 1) economic performances, 2) dynamic analysis of the party identification, 3) leader characteristics. Brief review in De Sio (2011, p. 49-51).
[71] (De Sio, 2011, p. 57-8)
[72] (De Sio, 2011, p. 57)

probabilistic approach.

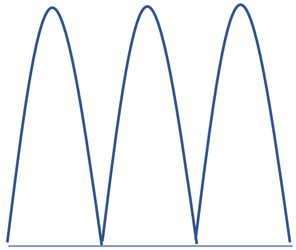
I have drawn some unimodal positional curves below, in two scenarios A and B: in scenario A, each of the three parties occupy exactly 1/3 of the positional space each, implying - probabilistically – that each elector necessarily votes for the nearest party without any uncertainty; in scenario B, each party covers more than 1/3 of the continuum, implying that the electors which are in a, b, c, and d identify themselves in the left-right positioning in which they could logically select to vote for another nearest party.

I can assume that the area included between a-b and c-d – highlighted in green - will primarily be an uncertainty area which, divided by the summation of all areas occupied by the three parties, can be reconducted to a non-positional competition. I then proceed to measure this intersection area, calling it "not pos. comp." having a domain [0,1] and conversely its complement (1 - "not pos. comp.") will be equal to a share of a positional competition.

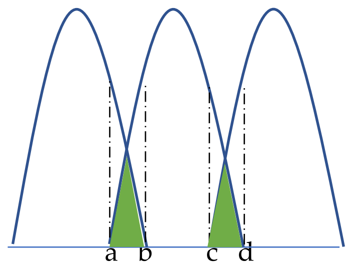
With regards to the majoritarian issues, these can be identified simply multiplying the first two kinds of competition, as they are in the middle by definition. This is a «virtual» and dummy variable obtained from the interaction of the first two, considered independently, since they are randomly present in the areas under the curves between a and b, and c and d.

In the example below it is possible to identify the positional competition equal to 1 in scenario A (figure 40), whereas in B (figure 41), this will surely be less then 1.

A *Figure 40 The positional competition equal to 1.*



B *Figure 41 The positional competition less than 1.*



I use the Beta functions introduced in the previous chapter to derive a new simplified model of permeability, directly obtained from the data distribution of all electors' positioning for each party, to identify a permeability for each party.

I perform a qualitative in-depth analysis of the four electoral systems into force, using data from 16 different elections from 1992 to 2018, to study the relation between n_2 and the dynamics of party competition. These 16 elections are an example of a tremendous dynamism of the party system both genetically, numerically and for the mechanic competition resulting from it. For these reasons, they can be considered as having ingenerated 16 electoral sub-systems.

The methodology used is mainly based on the logical models, here re-adapted in a qualitative way (Taagepera R. , 2005; 2008a; 2015), in the style of Taagepera's works since 1979 (Laakso-Taagepera, 1979; Taagepera - Shugart R.-M. , 1989; 1993; Taagepera R. , 2007b), also used in De Sio (April 2006, pp. 9-12; February 2008, pp. 220-7; 2011), which nowadays we need as an unavoidable epistemological base for the social sciences (Taagepera R. , 2017). These models will be tested empirically on the previously mentioned elections and data.

6.3. Directional tests

I now formalise and calculate the indicators introduced above, for each of the 16 elections from 1992 to 2018, by assembly. As the following empirical tests are founded on few cases, they are better referred to as "directional tests", recalling the logical methodology discussed in the introduction.

I introduce another variable measuring weighted ideological distance (px) between parties – defined in the interval [0,1]-, demonstrating and measuring the weight of the convergence hypothesis formulated by Downs (1957) and, indirectly, by Duverger (1951; 1954, pp. pp. 247, 269; 1955, p. p. 113). In fact, as we will see, the lesser the weighted ideological distance, the lesser the N, the more majoritarian the electoral system (in consideration of the first law of Duverger) and the lesser the weighted ideological distance. Px is defined by the formula:

$$px = \sum_{i=1}^{N_0} |\vartheta_i - \mu| s_i$$

Where $\mu = \sum_{i=1}^{N_0} \vartheta_i s_i$ being ϑ_i the mean positioning of the i-th party, s_i is the seats' share for the i-th party, μ represents the weighted mean of all ϑ_i values, with weighs s_i.

As formulated in the theoretical model of the new positional competition above, the empirical test needs to take into account both the positional and the non-positional competition variables.

I would like to test whether I can obtain an improved "effective" weighted ideological distance (px) by dividing by the weighted Effective number of parties N, which I will call N_{po}.

N_{po} is defined as the weighted average of N characterizing the different sections of the specific electoral system under analysis. In particular: with pure majoritarian or proportional electoral systems, N_{po} will be equal to the N calculated on the seats produced by those; conversely, in mixed electoral systems the share of seats allotted to the majoritarian section, the proportional one, and other types, will be calculated applying the weighted average of the different N to the seats allotted to each section and dividing by the sum of all seats.

Following my approach with px, I assume that N_{po} stabilizes positional competition; in fact, as seen above, the number of parties has a probabilistic impact on the summation of areas under the Beta curves of positional competition, net of intersected areas; therefore, dividing this by N_{po} neutralizes such impact. Being non-positional competition the complement to the positional one, it is stabilized in the same way.

In detail, N_{po} does not directly impact positional competition; however, it increases positional competition for 1<N_{po}<2, determining the anchor point N_{po}=1 in which the positional competition will be equal to 1, and a forbidden area logically defined under the line $y = -x + 2, for\ 1 < N_{po} < 2$. For these reasons, the impact of positional party competition is logically divided by N_{po}.

Another important variable to consider is the interaction between weighted ideological distance (px) and positional competition.
Table 10 shows the indicators introduced above, calculated for each of the 16 elections from 1992 to 2018, by assembly.

Table 10 Positional and non-positional party competition's variables, Italian elections 1992-2018

Election	Assembly	px	Non-pos. comp.	Pos. comp.	Pos. Comp./N_{po}	px*Pos.co/ N_{po}
1992	Chamber	0,723	0,417	0,583	0,088	0,064
1992	Senate	0,712	0,457	0,543	0,077	0,055
1994	Chamber	0,529	0,684	0,316	0,078	0,041
1994	Senate	0,511	0,559	0,441	0,141	0,072
1996	Chamber	0,495	0,535	0,465	0,113	0,056
1996	Senate	0,502	0,023	0,977	0,304	0,153
2001	Chamber	0,547	0,554	0,446	0,128	0,070
2001	Senate	0,548	0,480	0,520	0,175	0,096
2006	Chamber	0,468	0,453	0,547	0,271	0,127
2006	Senate	0,429	0,401	0,599	0,295	0,126
2008	Chamber	0,597	0,210	0,790	0,288	0,172
2008	Senate	0,583	0,240	0,760	0,284	0,165
2013	Chamber	0,601	0,258	0,742	0,185	0,111
2013	Senate	0,496	0,256	0,744	0,193	0,096
2018	Chamber	0,601	0,736	0,264	0,059	0,036
2018	Senate	0,645	0,712	0,288	0,065	0,042

I now proceed with the first step towards testing the impact of the weighted Effective number of parties (N_{po}) on weighted ideological distance (px), by assessing the relation between the two. This relation indicates how the weighted ideological distance is undoubtedly positively correlated to the weighted Effective number of parties N_{po}; for all passages see the appendix 6.1.

This is an encouraging result, although the application of N_{po} as a control variable in the final model still needs to be validated. The final aim is to obtain a model which has n as dependent variable and N_{po} as control variable.

I move to consider an alternative measure of weighted ideological distance, expressing it in terms of Euclidean distance (Abadir - Magnus, 2005, p. 1-3), also used by Adams et al. (2001, p. 17,22,31). I define this as pxq, which for the specific case will be equal to:

$$pxq = \sum_{i=1}^{N_0} (\vartheta_i - \mu)^2 s_i$$

I can now test all hypothesis defining weighted ideological distance (px and pxq) and the impact of the weighted Effective number of parties (N_{po}) on weighted ideological distance, summarized in appendix 6.2 (Table 34). I use both the F test and the $R^2 s$ to compare the relative strength of my hypotheses.

Comparing the px/N_{po} and px models as independent variables of n (models 1 and 2 of the appendix 6.2), I can consider the application of N_{po} as a control variable, taking n into consideration: variable n better explains weighted ideological distance px than it does px/N_{po}. I can therefore exclude the impact of N_{po} on weighted ideological distance and I can move to compare px (Model 2) and pxq (Model 3 in the appendix 6.2) to see which of the two is better explained by n.

Model 2 shows how n influences the linear weighted positional competition px with a better fitting than on pxq (as shown by Model 3). This also implies that it is not necessary to further investigate pxq in function of N_{po}.

To complement this analysis, I have produced two scatter plots to give a visual representation of px/N_{po} (Figure 42) and px expressed in terms of n (Figure 43).

Figure 42 Positional competition on weighted N (px/N_{po}) expressed in terms of n

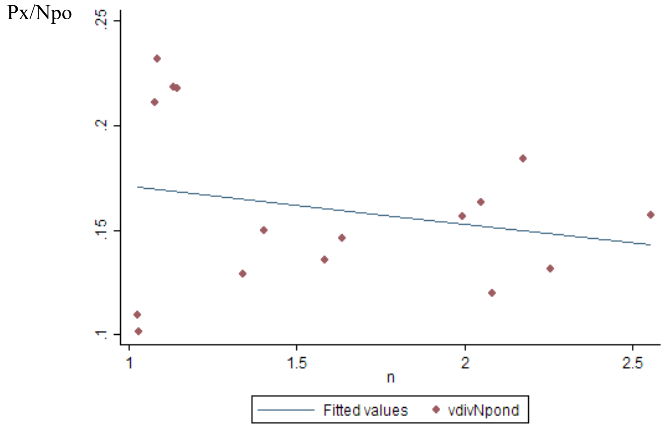
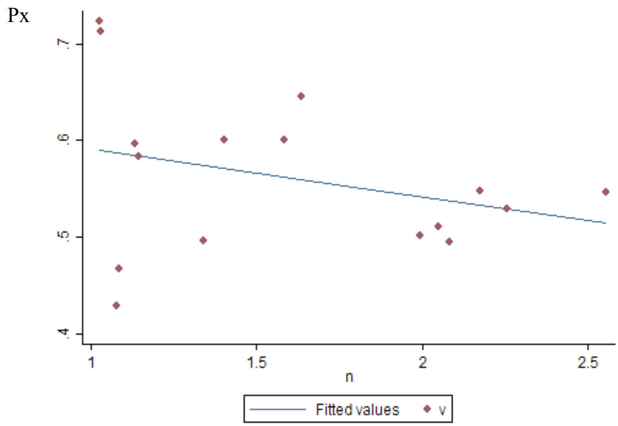


Figure 43 Positional competition (px) expressed in terms of n.



These graphs show how the intervention of N_{po} effectively smoothens the correlation with n. Moreover, px variance is smaller for higher values of n, and much higher for smaller values of n, confirming and expanding the aforementioned theory on electoral systems and party competition (mainly debated by Sartori), such that: electoral systems cannot completely create the type of party system, but they can only generate convergence, hence a reduction of the Effective number of parties for majoritarian electoral systems (explained by the previous demonstrated correlation). This suggestion will be better configured and consolidated later.

Although I had suggested to exclude quadratic weighted ideological distance (pxq), as previously shown in appendix 6.2, I have now taken the opportunity to investigate it in more thorough models in function of n (as a dependent variable), for completeness. This analysis suggests that pxq creates less explicative models, confirming the previous hypotheses; see appendix 6.3 for these tests.

Going back to the consideration of N_{po} as control variable, as done for pxq, I have tested the variable $\frac{px}{N_{po}}$ in more thorough models in function of n (as a dependent variable), for completeness; see appendix 6.3 for these tests. Also in this case, I confirm what previously shown in appendix 6.2, that N_{po} does not have a positive impact on px, but rather on positional competition (Pos. comp.$/N_{po}$). The politological significance of the fact that N_{po} does not add value to the model (weighted ideological distance will not be improved by dividing by N_{po}) is that weighted ideological distance already takes into account a measurement of parties, due to the presence of the previous weights s_i.

The meaning of (Pos. comp.$/N_{po}$) is the Effective positional competition, which is conditional to the non-positional one (n Pos. comp.). The latter is therefore equal to: $n\ Pos.comp. = N_{po} * \beta_{\frac{Pos.\ comp.}{N_{po}}} * (1 - Pos.comp.)$.

The joint model – model seven (in appendix 6.3) – can be discarded as well, when considering the significance of the constant and the other general parameters of analysis previously mentioned.

The model which best explains n in function of the other variables seems to be Model 8 (in appendix 6.3): the significance is at least 95% for each variable, all the other general parameters are better than other models and the explicated variance is the highest.

I can finally calculate the impact of each variable on n through the Beta correlation coefficients in base 100. Each unit variation of n is explained for 47% by Pos. comp.$/ N_{po}$, for 35% by px*Pos. comp.$/N_{po}$ and for 18% by the weighted ideological distance, due to the classical joint hypothesis of convergence by Downs and Duverger (referring to the implications of his first law); appendix 6.3 shows these results in the last column.

The meaning of px*Pos. comp./N_{po} is the degree of bi-polarization of the party system.

Figure 44 below represents the scatter plot of n explained though the "predicted values", applying Model 8.

Since majoritarian competition is the product between positional competition and its complement (non-positional competition), which have been measured above, in order to consider the majoritarian competition, the final econometric model must reattribute the Beta coefficients in base 100, considering this new coefficient in addition to the others in Model 8.

Figure 44 Scatter plot of the best model's fitting of n expressed in function of: weighted ideological distance (px), positional party competition (pos. comp.) and the weighted Effective number of parties (N_{po}). Real n on fitted values.

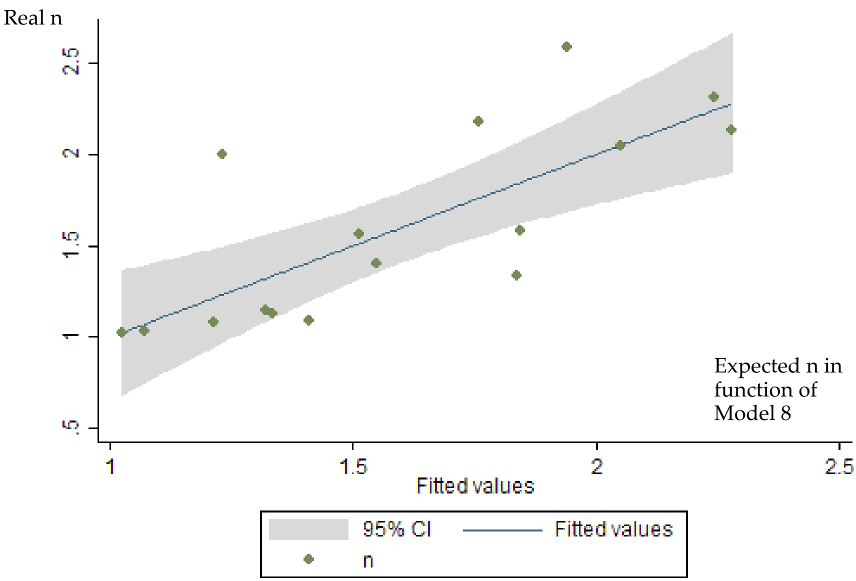


Table 36 in appendix 6.4, shows positional competition in function of n, thus switching the independent variable with the dependent one. This has been made possible by grouping the three independent variables using the predicted values from Model 8 - turning them into the dependent variable predicted n - and using n as the independent one.

I can finally obtain an equation with only two independent variables N_{po} and Pos. comp., using the coefficients for the correlation between px and N_{po} shown in Table 8 above, as well as the coefficients in Model 8. Replacing variable px with the correlation $px = 0.04778 * N_{po} + 0.380$, I obtain:

$$n = -9.008 * \left(0.0478 * N_{po} + 0.380\right) - 21.55 * \text{Pos. comp.}/N_{po} + 32.26 \\ * ((0.0478 * N_{po} + 0.380) * \text{Pos. comp.}/N_{po})) + 7.368)$$

I must consider the following logical constraints[73]: $N_{po} > 1$ by definition (as introduced in the first paragraph) and $0 < \text{Pos. comp.} < 1$, also by definition, resulting in the following final simplified system:

$$\begin{cases} n = 3.944 + (1 - \text{Pos. comp.})\left(1.542 - \dfrac{9.287}{N_{po}}\right) - 0.4304 * N_{po} \\ N_{po} > 1;\ 0 < \text{Pos. comp.} < 1; n > 1 \end{cases}$$

Figure 45 below draws the contour plot of n in function of N_{po} and Pos. comp., including the impossible values for n<1, to better understand the curve shape. Figure 46 draws the same relation, with n>1, N>1, listing the anchor[74] and noteworthy points afterwards.

[73] As defined by Taagepera and Shugart (Taagepera R. , 2008a, p. 34-50, 95-110) (Shugart - Taagepera, 2017, p. 11-13).
[74] See also Taagepera (2008a, p. 34-50, 95-110).

Figure 45 Contour plot of n in function of the weighted Effective number of parties N_{po} and the positional competition (pos. comp.). $N_{po} > 0$.

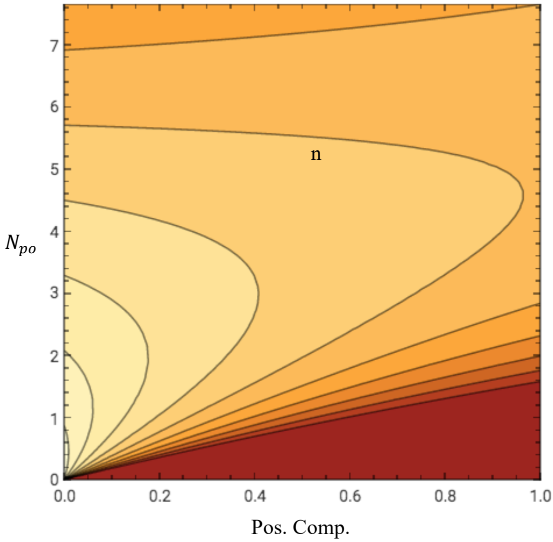
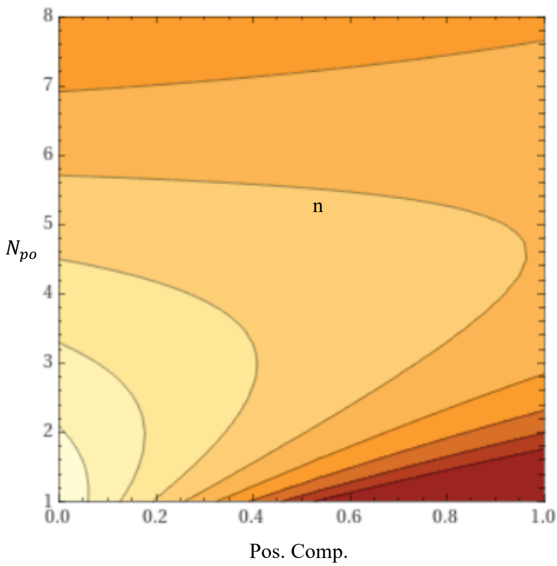


Figure 46 Contour plot of n in function of the weighted Effective number of parties N_{po} and the positional competition (pos. comp.). $N_{po} > 1$.



Noteworthy and Anchor Points:

N_{po} = 6.02	Break-even point: Pos. comp. constant for variation of n
N_{po} = 6.84	Max N_{po} for Pos. comp. = 0 (and n=1)
N_{po} = 7.57	Max N_{po} for Pos. comp. = 1 (and n=1)
N_{po} = 2.83	Max for Pos. comp. = 1 and n = 1, for other values n =1, the positional competition goes decreasing.
Pos. comp. = 0.32	Max value of Pos. comp. for N_{po} = 1 and n = 1.
n = 3.51	3.51 represents the maximum value of n with Pos. comp. = 0 and N_{po} = 1

The anchor point n=3.51 is in line with the values included in the data sample, as 87.75% of them are lower than or equal to this point. An empirical evidence supporting this finding is represented by the ideal-typical majoritarian electoral system - FPTP – of the Caribbeans, in which 3.5<n<4, as analyzed in detail by Nohlen (1993).

In any case, I remark how the results here shown can be considered only a qualitative point of view, a draft for directional relations among the positional, institutional and political variables. It is desirable to continue this research expanding the sample, aiming to obtain results that can be considered statistically significant and solid.

Chapter 7

Swing Vote and Downsian competition

7.1. Introduction

This chapter proposes a new method to understand how the electors change their vote preferences from one party to another and from an election to the next through electoral flows. In conjunction with the previous positional competition analysis, this method can be helpful to check if the intended ideological positioning of parties has produced the expected effects, if these have attracted electors from a competing party, and if they have left shares of electors belonging to the unwanted ideological area.

I suggest an alternative method to aggregate electoral flows. Goodman (1953) was the first to formalize a method to estimate the "swing votes", which is the number of voters moving from one election to another from one party to another, applying a simple regression model using territorial sub-units. Unfortunately, this method can produce, for inbound or outbound coefficients, either some negative coefficients or an unreasonable sum (greater than 1), or both, because the votes received by each party are at least 0 and at most equal to 1 (100%). Other methods (King, 1997; King - Rosen - Tanner, 1999, 1); (De Sio, 2009,1), may solve this problem, however they are very complex to use and need complex macros to work. For these reasons, I am suggesting a new method called "of mixture", which overcomes Goodman's problematics and it is much simpler than all existing methods.

Practically, in presence of matrices of multiple columns or rows (or both), representing the votes for each party from an election to another, this new method doubles the relative compatible ones and replicates these in proportion of the votes of the electoral results. Considering the matrices showing the source of votes for a party (inbound votes) for a given election, it is necessary to operate on rows, transforming their values in base 100; conversely, if the votes are outbound, I operate on columns where their values are also expressed in base 100.

The new logical method "of mixture" obtains a standard error of 0.62%, calculated on the sum of the squares of the differences between the values of the estimated (TC) and effective (TT) row values, divided by the number of rows; this is higher than King and others' at 0.37%, but lower than Goodman's at 0.93%[75]. This result of 0.62% is acceptable both: 1) because its equivalent absolute error 8.06%[76], calculated for the 2018 elections, is lower than the acceptability threshold of 15% set by Corbetta Parisi and Schadee (Corbetta, 1988)[77], 2) in terms of trade-offs of the criteria exposed by De Sio (2008, p. 84-90) - extremely easy to calculate, replicability and having an acceptable and contained error -, 3) since the manipulation of the matrices has an accuracy of 99.4%, according to their mathematical properties (Abadir - Magnus, 2005).

I investigate whether electoral vote flows can be a proxy for ideological positioning movements. I re-elaborated data from the electoral vote flow matrices of the elections from 1994 to 2013 (Bartolini - Chiaramonte - D'alimonte, 2002) (Schaade - Segatti, 2003) (D'Alimonte R. - Chiaramonte A., 2010) (De Sio - Paparo, 2014) and I subsequently crossed these with the ideological positioning estimated through the post-election surveys offered by ITANES (1948 - 2013) in the same period. The result is positive, obtaining an $R^2 adj. = 87.2\%$ between the expected ideological positioning and the actual one.

To sum up, I was able to reliably estimate the ideological party positioning for each party for the 2018 elections, using the existing ITANES party positioning data for the 2013 elections, calculating the relative expected values and modes to obtain the Beta functions by party (formalized in chapter 5), and finally interpolating these values with the 2013-2018 final flows matrix available, obtaining the final Beta functions. Applying the same methodology, I was also able to estimate the ideological party positioning for each party for the 1992 elections, using electoral flow and ideological positioning data from the 1994 elections.

7.2. For a new algebraic matrix of electoral flows with the Mixture method

I intend to build the electoral flow matrices with a completely algebraic approach, which I will call Mixture method. Putting to system the survey data from SWG (SWG, 2018), Ipsos (IPSOS, 2018) and CISE pre-electoral data (Carrieri, 2018), I am able to obtain a final matrix with lower errors than all the above-mentioned sources.

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[75] Calculation resulting from my analysis on an application of the model reported by De Sio (2009.1, p. 25).
[76] The value is calculated as 0.62% (standard error per line) * 13 (number of rows)
[77] They introduce the Vr indicator, which counts all the "impossible" coefficients because of the negative ones present in Goodman's flow matrix, placing the comprehensive tolerance threshold at 15%.

The principle is to aggregate as much information as possible, assuming that there are always some voting flows between one party and another, as also assumed in the HMD method of G. King and others (1997; G., 1999, 1), which although is currently the most complex, it is also the most accurate one with the lowest percentage of error (De Sio, 2009,1)[78] at 0.37%. This compares favorably to the Goodman's method, where the average percentage of error is 0.93%, calculated empirically[79].

In order to calculate the above-mentioned flow matrices percentage of error, as well as those which will follow, I consider the standard error as the squares' sum of the differences between values of the estimated (TC) and effectives (TT) row values, divided by the number of rows.

In presence of matrices of multiple columns or rows (or both), representing the votes for each party from an election to another, my proposed method consists in doubling the relative compatible ones and replicating these in proportion of the votes of the electoral results. Considering the matrices showing the source of votes for a party (inbound votes) for a given election, it is necessary to operate on rows, transforming their values in base 100; conversely, if the votes are outbound, I operate on columns where their values are also expressed in base 100.

When a matrix cell, row or column are empty, it could be due to a given party not been measured at all, or been considered grouped with others, or considered in coalition, or lastly due to the negative coefficients of flows estimation, as we will see.

Starting from the CISE matrix (Carrieri, 2018), shown in tables 11 and 12, with an error of 3.38%, and the SWG-Ipsos[80] matrix, shown in tables 13 and 14, with an error of 1.99%, the matrix obtained with the Mixture method, shown in tables 15 and 16, has an error of 0.62%, which is higher than both the Eidis and HMD methods, but lower than Goodman's methodology.

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[78] Although it is presented from the author himself as the method with the lowest error "between 0.01 and 0.02" (ibid., p. 28), comparing the flow matrices with the Eidis and HMD methods – by calculating the sum of the squares of the differences between the estimated (TC) and actual (TT) row values, and dividing by the number of rows - there is a median error per line equal to 0.364% for the first method and 0.37% for the second, almost equal.
[79] Calculation resulting from my analysis on an application of the model reported by De Sio (2009.1, p. 25).
[80] Presented in aggregate form using the method just introduced.

This result of 0.62% is acceptable both: 1) because its equivalent absolute error 8.06%[81], calculated for the 2018 elections, is lower than the acceptability threshold of 15% set by Corbetta Parisi and Schadee (Corbetta, 1988)[82], 2) in terms of trade-offs of the criteria exposed by De Sio (2008, p. 84-90) - extremely easy to calculate, replicability and having an acceptable and contained error -, 3) since the manipulation of the matrices has an accuracy of 99.4%, according to their mathematical properties (Abadir - Magnus, 2005).

Table 11 The "inbound votes" flow matrix of the 2018 Italian general elections (made 100 the sum of their relative parties' votes), showing inbound votes from the 2013 elections (expressed in terms of vote percentages), obtained from the new Mixture method; the 2013 columns include other parties. Source: CISE (Carrieri, 2018)

	2013	Ingroia	Coalition Bersani	Coalition Monti	M5s	PDL + FDI	Other parties	18-22 years	Don't vote
2018	Potere al Popolo	0.26	0.22	0.01	0.14	0.02	0.60	0.09	0.13
	LeU	0.40	2.22	0.33	0.30	0.02	0.14	0.23	0.57
	Pd	0.21	11.02	3.15	0.56	0.43	0.88	1.05	2.30
	+ Europa	0.08	0.89	0.15	0.14	0.03	0.46	0.16	0.52
	Insieme	0.01	0.18	0.00	0.07	0.02	0.00	0.06	0.08
	Civica Popolare	0.01	0.10	0.33	0.05	0.00	0.00	0.02	0.05
	M5s	0.22	2.48	0.93	12.39	1.20	0.13	1.37	4.22
	Noi con l'Italia -UDC	0.00	0.08	0.15	0.03	0.11	0.00	0.08	0.05
	Fi	0.10	0.46	0.35	0.49	6.74	0.31	0.62	1.91
	Lega	0.03	0.58	0.54	1.59	5.29	0.51	0.53	2.20
	FdI	0.04	0.20	0.43	0.26	1.40	0.92	0.30	0.49
	Other	0.01	0.04	0.09	0.09	0.08	1.52	0.11	0.23
	Don't vote	0.16	1.67	0.73	1.32	0.72	0.37	1.32	13.14

[81] The value is calculated as 0.62% (standard error per line) * 13 (number of rows)
[82] They introduce the Vr indicator, which counts all the "impossible" coefficients because of the negative ones present in Goodman's flow matrix, placing the comprehensive tolerance threshold at 15%.

Table 12 The totals "inbound votes" by 2018 party and their discards from the real ones, obtained from the flow matrix in Table 11.

	2013	EXP 2018	Real %	Quadratic discards
2018	Potere al Popolo	1.46	0.80	0.44
	LeU	4.19	2.40	3.23
	Pd	19.60	13.19	41.13
	+ Europa	2.43	1.80	0.40
	Insieme	0.42	0.42	0.00
	Civica Popolare	0.57	0.38	0.03
	M5s	22.94	23.07	0.02
	Noi con l'Italia -UDC	0.51	0.92	0.17
	Fi	10.98	9.87	1.23
	Lega	11.27	12.24	0.94
	FdI	4.04	3.07	0.95
	Other	2.18	2.43	0.06
	Don't vote	19.44	29.42	99.52
	Tot	100.0	100.0	**3.38**

Table 13 The "inbound votes" flow matrix of the 2018 Italian general elections, showing inbound votes from the 2013 elections (expressed in terms of vote percentages), obtained from the new Mixture method; the 2013 columns exclude other parties. Data Source: (SWG, 2018); (IPSOS, 2018), my elaboration.

	Votes [0,1]	**0.18**	**0.02**	**0.16**	**0.03**	**0.18**	**0.07**	**0.02**	**0.01**	**0.27**	**0.06**
	2013	PD + CD	Sel	Pdl	Lega	M5s	Mountains	Ingroia	Fare	Abstained	18-22 years
2018	Pd	8.64	0.86	0.38	0.05	0.37	2.30	0.08	0.04	1.00	0.67
	+ Europa	0.96	0.10	0.00	0.00	0.22	0.49	0.05	0.02	0.00	0.24
	Insieme-Civica Popolare	0.26	0.03	0.00	0.00	0.06	0.26	0.02	0.01	0.21	0.12
	LeU	0.99	0.50	0.00	0.00	0.09	0.16	0.22	0.06	0.32	0.18
	Fi	0.42	0.04	6.54	0.83	0.59	0.86	0.03	0.01	1.05	0.43
	Ln	0.53	0.04	5.79	1.49	1.10	0.63	0.12	0.06	3.01	0.61
	FdI	0.16	0.02	1.75	0.22	0.14	0.64	0.02	0.01	0.26	0.06
	M5s	2.65	0.27	1.33	0.19	13.96	0.79	0.25	0.16	3.93	1.58
	Other	1.15	0.12	0.36	0.05	0.36	0.47	0.62	0.32	0.63	0.06
	Abstained	2.73	0.27	0.35	0.04	1.04	0.81	0.17	0.09	16.19	2.13
		18.48	2.24	16.50	2.87	17.93	7.41	1.58	0.79	26.61	6.09

Table 14 The total "inbound votes" by 2018 party and their discards from the real ones, obtained from the flow matrix in Table 13.

	Votes [0,1]		**1.00**		
		2013	Exp 2018	Real %	Quadratic discards
2018	Pd		14.39	13.19	1.42
	+ Europa		2.08	1.80	0.08
	Insieme-Civica Popolare		0.97	0.42	0.30
	LeU		2.52	2.40	0.02
	Fi		10.80	9.87	0.86
	Ln		13.38	12.24	1.31
	FdI		3.27	3.07	0.04
	M5s		25.13	23.07	4.25
	Other		4.13	4.53	0.16
	Abstained		23.82	29.42	31.36
	Total		100.49	100.00	**1.99**

Table 15 The "inbound votes" flow matrix of the 2018 Italian general elections, showing inbound votes from the 2013 elections (expressed in terms of vote percentages), obtained from the new Mixture method; the 2013 columns include other parties. Data Source: (SWG, 2018); (IPSOS, 2018), my elaboration.

	votes	**0.18**	**0.02**	**0.16**	**0.03**	**0.17**	**0.07**	**0.022**	**0.01**	**0.26**	**0.06**	**0.03**
	2013	PD + SVP + CD (Bersani)	Sel	PDL (FI) +FdI	Lega Nord	M5s	CL. Monti	Ric. Civ. Ingroia	Fare	Do not vote	18-22 years	Other parties
2018	Potere al Popolo	0.11	0.01	0.01	0.00	0.08	0.01	0.17	0.00	0.06	0.05	0.20
	LeU	0.99	0.49	0.01	0.00	0.09	0.16	0.25	0.08	0.26	0.16	0.05
	Pd	8.23	0.82	0.37	0.05	0.35	2.20	0.09	0.05	0.79	0.57	0.35
	+ Europa	0.64	0.06	0.02	0.00	0.14	0.32	0.04	0.02	0.24	0.14	0.15
	Insieme	0.16	0.01	0.03	0.00	0.03	0.00	0.01	0.01	0.10	0.08	0.00
	Civica Popolare	0.07	0.01	0.00	0.00	0.02	0.23	0.01	0.00	0.05	0.02	0.00
	M5s	2.61	0.27	1.35	0.19	13.51	0.78	0.28	0.20	3.21	1.39	0.08
	Noi con l'Italia-UDC	0.14	0.02	0.19	0.03	0.07	0.30	0.00	0.00	0.08	0.13	0.00
	Fi	0.40	0.04	6.42	0.81	0.55	0.83	0.03	0.02	0.83	0.36	0.17
	Lega	0.50	0.04	5.58	1.43	1.01	0.60	0.13	0.08	2.35	0.51	0.33
	FdI	0.13	0.01	1.47	0.19	0.12	0.52	0.02	0.01	0.17	0.04	0.34
	Other	0.43	0.04	0.14	0.02	0.13	0.18	0.26	0.15	0.20	0.02	0.59
	Don't vote	3.57	0.36	0.47	0.06	1.33	1.07	0.25	0.14	17.52	2.47	0.34
	Total	17.97	2.18	16.04	2.79	17.43	7.20	1.53	0.76	25.87	5.93	2.58

Table 16 The total "inbound votes" by 2018 party and their discards from the real ones, obtained from the flow matrix in Table 15.

	votes	**1.00**		
	2013	Exp 2018	Real %	Quadratic discards
2018	Potere al Popolo	0.70	0.80	0.01
	LeU	2.53	2.40	0.02
	Pd	13.86	13.19	0.44
	+ Europa	1.77	1.80	0.00
	Insieme	0.43	0.42	0.00
	Civica Popolare	0.40	0.38	0.00
	M5s	23.87	23.07	0.65
	Noi con l'Italia-UDC	0.97	0.92	0.00
	Fi	10.47	9.87	0.35
	Lega	12.56	12.24	0.10
	FdI	3.02	3.07	0.00
	Other	2.15	2.43	0.08
	Don't vote	27.58	29.42	3.38
	Total	100.29	100.00	**0.62**

I now calculate the outbound votes for each party for the 2013 elections by dividing each cell by its column margin from table 15 and multiplying by 100. The results are shown in tables 17 and 18.

Table 17 The "outbound votes" flow matrix of the 2013 Italian general elections (in base 100), showing outbound votes towards the 2018 election parties, obtained from the new Mixture method. Data Source: (SWG, 2018); (IPSOS, 2018), my elaboration.

	Ratings	**0.18**	**0.02**	**0.16**	**0.03**	**0.17**	**0.07**	**0.022**	**0.01**
	2013	PD + SVP + CD (Bersani)	Sel	PDL(FI+FdI) Berlusconi	Lega	M5s	Coalition. Monti	Ingroia	Fare
2018	Potere al Popolo	0.6	0.63	0.05	0.05	0.46	0.12	11.04	0.00
	LeU	5.50	22.44	0.05	0.05	0.50	2.20	16.10	10.04
	Pd	45.80	37.57	2.30	1.67	2.00	30.61	5.77	6.66
	+ Europa	3.54	2.86	0.10	0.10	0.82	4.47	2.36	2.73
	Insieme	0.89	0.53	0.18	0.18	0.19	0.00	0.64	0.68
	Civica Popolare	0.37	0.49	0.00	0.00	0.10	3.20	0.48	0.62
	M5s	14.53	12.29	8.39	6.98	77.52	10.90	18.07	26.73
	Noi con l'Italia - Udc	0.78	0.77	1.20	1.19	0.38	4.14	0.00	0.00
	Fi	2.25	1.85	40.01	29.07	3.17	11.55	2.03	2.35
	Lega	2.76	1.76	34.81	51.32	5.82	8.29	8.71	10.05
	FdI	0.71	0.58	9.14	6.64	0.66	7.26	1.28	1.47
	Other	2.38	1.96	0.86	0.62	0.75	2.45	17.22	19.87
	Don't vote	19.85	16.28	2.92	2.12	7.63	14.82	16.30	18.81
	Total	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

Table 18 The "outbound votes" flow matrix (second part) of the 2013 Italian general elections (in base 100), showing outbound votes towards the 2018 election parties, obtained from the new Mixture method; the 2013 columns include other parties. Data Source: (SWG, 2018); (IPSOS, 2018), my elaboration.

	Ratings		**0.26**	**0.06**	**0.03**
		2013	Don't vote	18-22 years	Other parties
2018	Potere al Popolo		0.24	0.77	7.61
	LeU		1.02	2.69	1.84
	Pd		3.04	9.55	13.46
	+ Europa		0.92	2.39	5.68
	Insieme		0.39	1.27	0.00
	Civica Popolare		0.19	0.28	0.00
	M5s		12.40	23.34	3.06
	Noi con l'Italia - Udc		0.32	2.26	0.00
	Fi		3.21	6.11	6.42
	Lega		9.09	8.58	12.64
	FdI		0.67	0.74	13.31
	Other		0.76	0.34	22.78
	Don't vote		67.74	41.68	13.18
	Total		100.00	100.00	100.00

How to read these matrices? Table 17 shows how the Forza Italia party has the most disloyal voters from the 2013 elections towards the 2018 ones, with only 40% of reconfirmed votes, and 34.81% of votes switching to Lega and 9.14% to FdI. The PD party follows suite, with only 45.8% of reconfirmed votes. Conversely, the most loyal voters are those of the M5S party, with 77.52% of reconfirmed votes, and only 5.82% switching to Lega, and 3.17% to Forza Italia.

It is difficult to draw firm considerations regarding FdI and LEU, since the first merged with Forza Italia in 2013 and the latter was newly constituted in 2018.

The most interesting matrix coefficients in Tables 17 and 18, from a politological perspective, refer to the PD party: 1) 14.53% of those who voted the PD coalition in 2013 switched to M5S in 2018, 2) 19.85% opted for the non-vote instead - the highest abstention data among all parties – , and 3) 5.5% switched to LEU.

Among the parties that no longer existed after the 2013 elections, the major outflows of votes have been directed towards the PD, voted in 2018 by 37.57% of the voters of Sel in 2013 and by the 30.61% of the voters of Monti, also from 2013. Symmetrically, the M5S was the most voted party in 2018 by 26.73% of former voters of "Fare" (by Oscar Giannino) from 2013, by 23.34% of under-22s, by 18.07% of former voters of RC (Ingroia) and by 12.4% of previously abstained voters.

I now calculate the inbound votes for each party for the 2018 elections by simply proportioning the values in Table 15 to make the row margins equal to 100. I obtain tables 19 and 20 below, which represents the composition of votes for each party in 2018, in relation to parties and groups from 2013.

Table 19 The "inbound votes" flow matrix of the 2018 Italian general elections (in base 100), showing inbound votes from the 2013 election parties, obtained from the new Mixture method. Data Source: (SWG, 2018); (IPSOS, 2018), my elaboration.

	Ratings	0.18	0.02	0.16	0.03	0.17	0.07
	2013	Coalition Bersani	Sel	PDL (FI) +FdI	Lega	M5s	CL. Monti
2018	Potere al Popolo	16.40	1.95	1.16	0.20	11.35	1.20
	LeU	39.09	19.37	0.33	0.06	3.47	6.26
	Pd	59.39	5.92	2.66	0.34	2.52	15.91
	+ Europa	36.04	3.53	0.89	0.15	8.08	18.23
	Insieme	37.26	2.70	6.55	1.14	7.58	0.00
	Civica Popolare	16.30	2.62	0.00	0.00	4.48	57.02
	M5s	10.94	1.12	5.64	0.82	56.60	3.29
	Noi con l'Italia Udc	14.59	1.74	19.87	3.45	6.94	30.88
	Fi	3.86	0.39	61.33	7.75	5.28	7.95
	Lega	3.95	0.31	44.46	11.40	8.08	4.76
	FdI	4.20	0.42	48.53	6.13	3.83	17.30
	Other	19.91	1.98	6.38	0.81	6.05	8.19
	Don't vote	12.93	1.29	1.70	0.21	4.82	3.87

Table 20 The "inbound votes" flow matrix (second part) of the 2018 Italian general elections (in base 100), showing inbound votes from the 2013 election parties, obtained from the new Mixture method; the 2013 columns include other parties. Data Source: (SWG, 2018); (IPSOS, 2018), my elaboration.

	Ratings	0.02	0.01	0.26	0.06	0.03	1.00
	2013	RC Ingroia	Fare	Don't vote	18-22 years	Other parties	Sum up
2018	Potere al Popolo	24.23	0.00	8.88	6.54	28.08	100.00
	LeU	9.77	3.03	10.41	6.31	1.88	100.00
	Pd	0.64	0.37	5.67	4.09	2.50	100.00
	+ Europa	2.05	1.18	13.54	8.02	8.29	100.00
	Insieme	2.30	1.20	23.70	17.58	0.00	100.00
	Civica Popolare	1.81	1.17	12.45	4.15	0.00	100.00
	M5s	1.16	0.86	13.44	5.80	0.33	100.00
	Noi con l'Italia Udc	0.00	0.00	8.68	13.85	0.00	100.00
	Fi	0.30	0.17	7.93	3.46	1.58	100.00
	Lega	1.06	0.61	18.72	4.05	2.60	100.00
	FdI	0.65	0.37	5.75	1.46	11.35	100.00
	Other	12.29	7.06	9.08	0.94	27.31	100.00
	Don't vote	0.91	0.52	63.55	8.97	1.23	100.00

I comment that the M5S base of voters in 2018 is represented for 56.6% by its previous voters from 2013, for 13.44% by abstained voters from 2013 and for 10.94% by former PD voters, also from 2013. Symmetrically, PD records the highest level of "electoral stillness" in 2018, with the highest percentage (59.39%) of voters coming from its previous voters from 2013 and, as second contributor, the former voters of Monti, with 15.91%.

It is also interesting to remark how the 2018 electoral composition of Lega only includes its own voters from 2013 for 11.4%, whilst the most significant component is represented by the former voters of the center-right coalition (FI and FdI) with 44.46% and previously abstained voters with 18.72%. The former Monti voters represent the most significant component of Civica Popolare and Noi con l'Italia. The most significant component of FdI comes from former FI and FdI coalition voters with

48.53%, surprisingly followed by former Monti's coalition voters with 17.3%, and by minor parties (named as "Other parties" in the tables) with 11,35%. Finally, the most significant component of new party Potere al Popolo comes mainly from minor parties with 28%, from RC with 24.23% and from PD with 15.4%.

7.3. The correlation between the electoral flow matrix and the ideological positioning on the left-right axis

I can now test the strong hypothesis that as voters move from one party to another, they retain an "average party ideological identity" that they carry with them when voting for another party at the next election. In other words, I want to ascertain whether electoral vote flows can be a proxy for ideological positioning movements.

To test this hypothesis, I use the following operationalization: made 100 the row margins of each party (relative to the next election), the summation by line of the products of the coefficients of this matrix and the values of the parties' ideological positioning of the preceding elections (placed on the same column), must be equal to the ideological positioning of each party as recorded by the ITANES survey data (1948 - 2013).

I re-elaborated the data from the electoral vote flow matrices of the elections from 1994 to 2013 (Bartolini - Chiaramonte - D'alimonte, 2002) (Schaade - Segatti, 2003) (D'Alimonte R. - Chiaramonte A., 2010) (De Sio - Paparo, 2014), making the appropriate modification for row margins, and I subsequently crossed these with the ideological positioning estimated through the post-election survey data offered by ITANES (1948 - 2013) in the same period, multiplying the coefficients relating to the above ITANES data for the expected value grouped for each party.

Table 21 summarizes the correlation between the real positional mean and the predicted one, obtained using the electoral flow data.

Table 21 The correlation between the electoral flow matrix and the ideological positioning on the left-right axis. The real positional mean is explained by the predicted positional mean

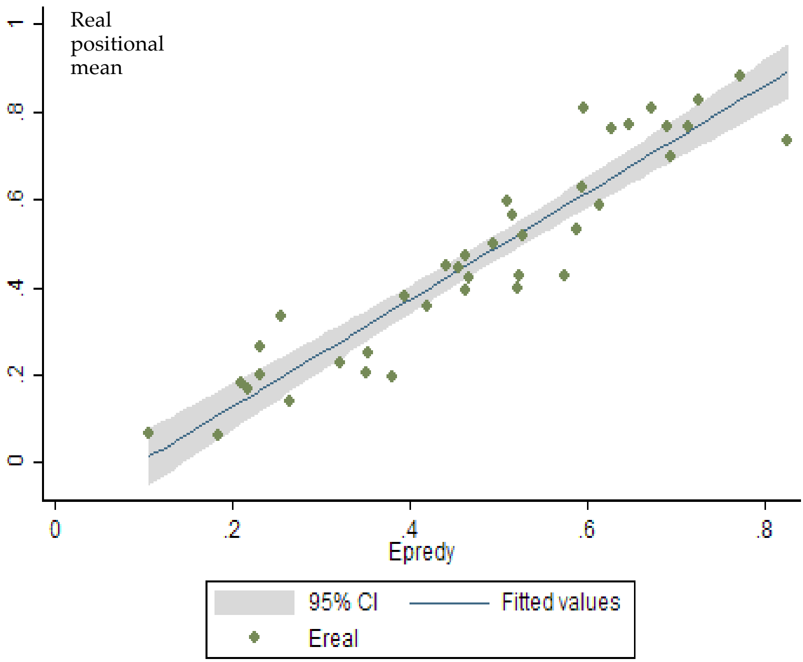
VARIABLES	1 Real Mean
Predicted Mean	1.219 ***
	(0.0757)
Constant	-0.115 ***
	(0.0386)
R^2	0.875
R^2 Adj	0.8716
Prob > F Test	0.0000
Observations	39
Root MSE	.0843

Standard Errors in Parentheses
p < 0.01, * * p < 0.05, * p < 0.1

The regression confirms the hypothesis with a significant R^2 adj. of 87.2% and statistical significance of the coefficient and the constant greater than 99%. The internal coherence of the model is at least 99.99% (F test). Finally, the Root MSE indicates an average standard error estimation of the ideological positioning for each party at only 8.43%, meaning that the estimation of the ideological positioning will be within the confidence interval of +-0.0843.

Figure 47 provides the relative scatter plot of the correlation presented in Table 21 for a confidence interval of 95%.

Figure 47 Scatter plot of the correlation between the electoral flow matrix and the ideological positioning on the left-right axis. The real positional mean is explained by the predicted positional mean (Epredy).



A second regression is necessary in order to construct the Beta probability density function for each party. I follow the same testing process as above, using the statistical mode values instead of the expected value of parties' ideological positioning of the preceding elections. Therefore, made 100 the row margins of each party (relative to the next election), I calculate the summation by line of the products of the coefficients of this matrix and the statistical mode values of the parties' ideological positioning of the preceding elections (placed on the same column), using the same ITANES survey data as used previously.

This will help me establish whether a correlation exists between moving voters and the average standard deviation of the Beta probability density function of the ideological positioning of each party. Table 22 summarizes the results of this statistical analysis.

Table 22 The correlation between the electoral flow matrix and the ideological positioning on the left-right axis. The real positional mode is explained by the predicted positional mode

VARIABLES	1 Real Mode
Predicted Mode	1.143 ***
	(0.107)
Constant	-0.0365
	(0.0574)
R^2	0.754
Adj R^2	0.7478
Prob > F Test	0.0000
Observations	39
Root MSE	.1639

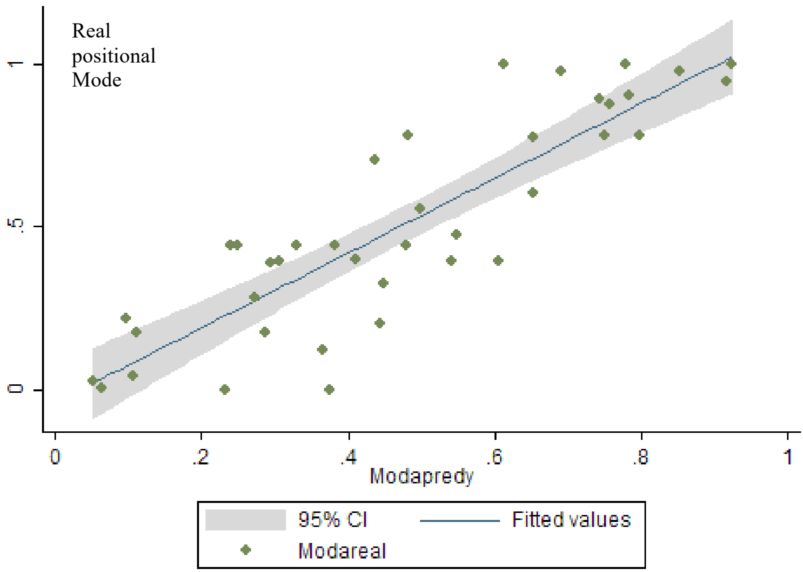
Standard Errors in Parentheses
p < 0.01, * * p < 0.05, * p < 0.1

Again, I notice the strong correlation indicated by an R^2 adj. equal to 74.8%, a significance of the estimated values greater than 99%. Unlike before, the significance of the constant is not greater than 99%, however this can be ignored since the constant is close to zero, and therefore very sensitive even to infinitive fluctuations; this is irrelevant also from a logical and statistical perspective, since to a predicted median ideological positioning value equal to 0 corresponds on average an equal real value of 0; therefore, the same is valid for the previous expected values.

Also in this case, the internal coherence of this models is above 99.99%. The Root MSE is certainly higher – 0.1639 – but mainly due to the minimum unit measure of the statistical mode measuring party ideological identification, which is on average equal to $0,\overline{1}$ (1/9), which directly impacts the model error.

Figure 48 provides the relative scatter plot of the correlation presented in Table 22 for a confidence interval of 95%.

Figure 48 Scatter plot of the correlation between the electoral flow matrix and the ideological positioning on the left-right axis. The real positional mode is explained by predicted positional mode (Modapredy).



The final conclusion of this analysis is that about 20% of electors, which do not collocate themselves of the left-right continuum, as said in Chapter 5 – calculated as the average complement to the R^2s presented in tables 21 and 22 - generate some random movements among party votes which also imply some ideological noise.

In conclusion, from the vote's flow matrixes and the parties' positioning in one election, I can summarize the most important sources of the bias of the whole model able to predict the beta functions of the party positioning for a following or previous election.

In order to predict positional beta functions it has been necessary to predict both mean and mode values using two respective models. Statistically, the mean of these beta functions is in the "central" area of the left-right continuum, in particular, when the mode is greater than 0.5 the mean is to the left of it, whereas if the mode is lower than 0.5 the mean is to the right of it.

Theoretically, the fact that the predicted mean has a higher R squared and a lower standard error (Root MSE) than the predicted mode, means that the final beta functions will have more solid anchor points in the central area of the left-right continuum, and a more unstable determination in the extreme points of the continuum. In particular, the standard error for the extreme points is about double of the central ones.

The reasons for these characteristics have been identified mainly in the following reasons:

1) The minimum unit measure of the statistical mode measuring party ideological identification is on average equal to $0,\overline{1}$ (1/9), which mostly impacts the mode's standard error which is 0.1639.

2) About 20% of electors do not collocate themselves on the left-right continuum, as said in Chapter 5.

Last but not least, the direction of the biases is also observable: in fact, it arises as the previous theory of party competition has shown in the previous chapter, confirming their validity. In particular, the presence of majoritarian electoral systems goes to increase the polarization towards the extreme ideological positions. Therefore, the more majoritarian the electoral system, the higher the standard error registered in correspondence of the modes - in correspondence of the external points of the left-right continuum - in particular, in this case, the predicted values tend to be more towards the center of the continuum compared to the real ones.

The power of the model is noteworthy, given the strong simplified model that does not include the electoral system's features as instead done in the previous chapter. A possible future perspective could – in addition to widening the sample size – include in both regression models (for mean and mode) the variable n which goes to synthetize the disproportionality of the electoral system, and a proxy of a comprehensive majoritarian degree of the electoral system in use.

7.4. The construction of ideological positioning for 2018 and 1992

I can now move to define the ideological positioning of each party in 2018 and 1992.

Regards the 2018, the first step is to produce a flow matrix, with all row margins equal to 100, obtained dividing the values in the inbound vote matrix by their row margins, like done in tables 19 and 20.

Before doing this, I will have to clean the outbound vote matrix in Table 18, excluding the 18-22 years column as this cannot be crossed with the survey data at our disposal. In order to exclude a row or column, the respective row or column margin must be equal to 100, such that it does not cross-influence the rows if you intervene on the columns – like in this case - and vice versa the columns if you intervene on the rows. Therefore, I can exclude the 18-22 years column in Table 18 since it is already in base 100. Such exclusion will determine a standard error increase that will be simply recalculated as in tables 15 and 16, and equal to 0.81, which is only 0.19 percentage points higher than the original one (0.62), still acceptable.

Tables 23 and 24 represents the inbound vote flows for each party, made 100 the total votes obtained by each party in 2018.

Table 23 Inbound vote flow matrix of the 2018 Italian general elections (in base 100), showing inbound votes from the 2013 election parties, obtained from the new Mixture method; the 2013 columns include other parties and exclude inbound votes from 18-22 year voters. Data Source: (SWG, 2018); (IPSOS, 2018), my elaboration.

	Ratings	0.19	0.02	0.17	0.03	0.18
	2013	Coalition Bersani	Sel	PdI (FI +FdI)	Lega	M5s
2018	Power to the People	22.81	2.72	1.61	0.28	15.79
	LeU	39.84	19.75	0.34	0.06	3.53
	Pd	60.92	6.07	2.73	0.34	2.58
	+ Europa	39.30	3.85	0.97	0.17	8.81
	Insieme	37.26	2.70	6.55	1.14	7.58
	Civica Popolare	16.30	2.62	0.00	0.00	4.48
	M5s	10.97	1.13	5.66	0.82	56.79
	Noi con l'Italia-UDC	14.59	1.74	19.87	3.45	6.94
	Fi	3.93	0.39	62.31	7.87	5.37
	Lega	4.06	0.31	45.65	11.70	8.30
	FdI	4.74	0.47	54.74	6.92	4.32
	Other parties	27.39	2.73	8.78	1.11	8.32
	Not Rating	13.09	1.30	1.72	0.22	4.88

Table 24 Inbound vote flow matrix (second part) of the 2018 Italian general elections (in base 100), showing inbound votes from the 2013 election parties, obtained from the new Mixture method; the 2013 columns include other parties and exclude inbound votes from 18-22 year voters. Data Source: (SWG, 2018); (IPSOS, 2018), my elaboration.

	Ratings	**0.08**	**0.02**	**0.01**	**0.27**	**0.03**	**1.00**
	2013	Coalition Monti	Ingroia	Fare	Don't vote	Other parties	Total raw
2018	Power to the People	1.67	33.69	0.00	12.35	9.09	100
	LeU	6.38	9.96	3.09	10.61	6.43	100
	Pd	16.32	0.66	0.38	5.82	4.20	100
	+ Europa	19.88	2.24	1.28	14.77	8.74	100
	Insieme	0.00	2.30	1.20	23.70	17.58	100
	Civica Popolare	57.02	1.81	1.17	12.45	4.15	100
	M5s	3.30	1.17	0.86	13.49	5.82	100
	Noi con l'Italia-UDC	30.88	0.00	0.00	8.68	13.85	100
	Fi	8.08	0.30	0.17	8.06	3.52	100
	Lega	4.88	1.09	0.63	19.21	4.16	100
	FdI	19.51	0.73	0.42	6.49	1.65	100
	Other parties	11.27	16.90	9.71	12.50	1.29	100
	Not Rating	3.92	0.92	0.53	64.34	9.08	100

I can now proceed to the last part of my analysis.

Firstly, I select the data from the 2013 ITANES survey sample indicating the self-collocation on the left-right continuum of each elector, grouped by party. This data is defined in the interval [0,1], where 0 indicates a far-left and 1 a far-right self-collocation. For ease, I display these values in the first row of table 25 below, labeled as "E".

Secondly, I multiply each coefficient in the previous tables 23 and 24 matrix by the value in row E - of table 25 - relative to the same column and I divide by 100. In table 26, the column "E sum" represents the summation by row of the coefficients thus calculated in table 25, and column "E sum corrected" is the result of the application of the regression coefficients referred to the expected values (predicted mean) of each party positioning in table 21 to the values in "E sum".

Hence, Table 26 shows the conclusive results of the predicted party collocations, obtained from the party positioning of the previous election (2013) by means of the inbound vote flow matrix. In fact, the parties' ideological collocation in the 2013 elections "hooks" to the vote movements from the 2013 to the 2018 elections represented in the inbound vote flow matrix; it can then follow the movements of the votes, producing a final ideological collocation into the 2018 election, as empirically demonstrated before.

Table 25 Predicted party placement - expected mean - in the 2018 Italian general elections, from the 2013 election party positioning by means of the inbound vote flow matrix.

E		**0.20**	**0.15**	**0.83**	**0.80**	**0.40**	**0.52**	**0.20**	**0.57**	**0.56**	**0.47**
	2013	PD + CD +Svp	Sel	PDL (FI) +FdI	Lega	M5s	CL. Monti	Ingroia	Fare	Abstained	Other parties
2018	Potere al Popolo	0.05	0.00	0.01	0.00	0.06	0.01	0.07	0.00	0.07	0.04
	LeU	0.08	0.03	0.00	0.00	0.01	0.03	0.02	0.02	0.06	0.03
	Pd	0.12	0.01	0.02	0.00	0.01	0.08	0.00	0.00	0.03	0.02
	+ Europa	0.08	0.01	0.01	0.00	0.04	0.10	0.00	0.01	0.08	0.04
	Insieme	0.08	0.00	0.05	0.01	0.03	0.00	0.00	0.01	0.13	0.08
	Civica Popolare	0.03	0.00	0.00	0.00	0.02	0.29	0.00	0.01	0.07	0.02
	M5s	0.02	0.00	0.05	0.01	0.23	0.02	0.00	0.00	0.08	0.03
	Noi con l'Italia-UDC	0.03	0.00	0.16	0.03	0.03	0.16	0.00	0.00	0.05	0.06
	Fi	0.01	0.00	0.51	0.06	0.02	0.04	0.00	0.00	0.05	0.02
	Lega	0.01	0.00	0.38	0.09	0.03	0.03	0.00	0.00	0.11	0.02
	FdI	0.01	0.00	0.45	0.06	0.02	0.10	0.00	0.00	0.04	0.01
	Other parties	0.06	0.00	0.07	0.01	0.03	0.06	0.03	0.06	0.07	0.01
	Not Rating	0.03	0.00	0.01	0.00	0.02	0.02	0.00	0.00	0.36	0.04

Table 26 The totals of predicted party placement by 2018 parties - expected mean - from the 2013 election party positioning by means of the inbound vote flow matrix.

E			
	2013	E sum	E sum corrected
2018	Potere al Popolo	0.32	0.27
	LeU	0.29	0.24
	Pd	0.31	0.26
	+ Europa	0.37	0.33
	Insieme	0.40	0.37
	Civica Popolare	0.45	0.43
	M5s	0.43	0.41
	Noi con l'Italia-UDC	0.52	0.52
	Fi	0.71	0.75
	Lega	0.67	0.70
	FdI	0.68	0.72
	Other parties	0.40	0.37
	Not Rating	0.49	0.49

Table 27 below follows the same methodology, using the statistical mode instead of the mean: row "E" in Table 25 becomes "Mode", still defined in the interval [0,1]; in Table 28, column "E sum corrected" becomes "Mode sum corrected", which is the result of the application of the regression coefficients referred to the predicted positional mode values of each party positioning in table 22.

The values in columns "E sum corrected" and "Mode sum corrected" provide the coefficients useful to graphically represent the Beta functions for all parties.

Table 27 Predicted party placement - expected mode - in the 2018 Italian general elections, from the 2013 election party positioning by means of the inbound vote flow matrix.

Mode		**0.21**	**0.05**	**0.89**	**0.90**	**0.40**	**0.48**	**0.04**	**0.56**	**0.56**	**0.44**
	2013	PD + CD +Svp	Sel	PDL (FI) +FdI	Lega	M5s	Coalition Monti	Ingroia	Fare	Abstained	Other parties
2018	Potere al Popolo	0.05	0.00	0.01	0.00	0.06	0.01	0.01	0.00	0.07	0.04
	LeU	0.08	0.01	0.00	0.00	0.01	0.03	0.00	0.02	0.06	0.03
	Pd	0.13	0.00	0.02	0.00	0.01	0.08	0.00	0.00	0.03	0.02
	+ Europa	0.08	0.00	0.01	0.00	0.04	0.10	0.00	0.01	0.08	0.04
	Insieme	0.08	0.00	0.06	0.01	0.03	0.00	0.00	0.01	0.13	0.08
	Civica Popolare	0.03	0.00	0.00	0.00	0.02	0.27	0.00	0.01	0.07	0.02
	M5s	0.02	0.00	0.05	0.01	0.23	0.02	0.00	0.00	0.07	0.03
	Noi con l'Italia-UDC	0.03	0.00	0.18	0.03	0.03	0.15	0.00	0.00	0.05	0.06
	Fi	0.01	0.00	0.55	0.07	0.02	0.04	0.00	0.00	0.04	0.02
	Lega	0.01	0.00	0.40	0.10	0.03	0.02	0.00	0.00	0.11	0.02
	FdI	0.01	0.00	0.48	0.06	0.02	0.09	0.00	0.00	0.04	0.01
	Other parties	0.06	0.00	0.08	0.01	0.03	0.05	0.01	0.05	0.07	0.01
	Not Rating	0.03	0.00	0.02	0.00	0.02	0.02	0.00	0.00	0.36	0.04

Table 28 The totals of predicted party placement by 2018 parties - expected mode - from the 2013 election party positioning by means of the inbound vote flow matrix.

Mode			
	2013	Mode sum	Mode sum
2018	Potere al Popolo	0.26	0.26
	LeU	0.25	0.25
	Pd	0.30	0.31
	+ Europa	0.35	0.37
	Insieme	0.40	0.42
	Civica Popolare	0.42	0.44
	M5s	0.43	0.45
	Noi con l'Italia-UDC	0.52	0.56
	Fi	0.75	0.82
	Lega	0.70	0.77
	FdI	0.71	0.78
	Other parties	0.37	0.39
	Not Rating	0.48	0.52

Applying the same methodologies, I was also able to estimate the expected (table 30) and mode (table 31) values of ideological party positioning for each party for the 1992 elections using electoral flow and ideological positioning data from the 1994 elections. I sourced the 1994 electoral flow data from the re-elaboration (table 29) of Diamanti and Mannheimer's matrix of flow (Milano a Roma, 1994, p. 114), and the 1994 ideological positioning data from the ITANES survey data used above.

Table 29 Inbound vote flow matrix of the 1992 Italian general elections (in base 100), showing inbound votes from the 1994 election parties. Source: Diamanti - Mannheimer (1994, p. 114), my re-elaboration.

	1994	Progressives	Patto	Polo	Others	
1992	RC	58	6	22	14	100
	PDS	74	4	15	7	100
	La Rete	46	15	28	11	100
	Verdi	26	21	41	12	100
	PSI	26	15	48	11	100
	PRI-PSDI-PLI	9	26	54	11	100[83]
	DC	3	61	30	6	100
	MSI	2	2	90	6	100

Table 30 Predicted party placement - expected mean - in the 1992 Italian general elections, from the 1994 election party positioning by means of the inbound vote flow matrix.

E (1992)		**0.17**	**0.51**	**0.71**	**0.43**	**E**	**E corrected**
	1994	Progressisti	Patto	Polo	Others		
1992	RC	11.27	1.47	18.23	2.81	0.34	0.30
	PDS	13.62	0.93	11.76	1.33	0.28	0.22
	La Rete	9.27	3.80	24.06	2.29	0.39	0.37
	Verdi	5.47	5.55	36.77	2.61	0.50	0.50
	PSI	5.20	3.77	40.93	2.27	0.52	0.52
	PRI-PSDI-PLI	1.96	7.15	49.30	2.43	0.61	0.63
	DC	0.83	21.22	35.38	1.71	0.59	0.61
	MSI	0.35	0.44	67.56	1.09	0.69	0.73

[83] The value shown in the original source appears as 110. I therefore re-weighted for 100.

Table 31 Predicted party placement - expected mode - in the 1992 Italian general elections, from the 1994 election party positioning by means of the inbound vote flow matrix

Mode (1992)		**0.07**	**0.51**	**0.80**	**0.08**	**Mode**	**Mode Corrected**
	1994	Progressisti	Patto	Polo	Others		
1992	RC	4.45	1.48	20.49	0.54	0.27	0.27
	PDS	5.37	0.93	13.22	0.26	0.20	0.19
	La Rete	3.66	3.84	27.04	0.44	0.35	0.36
	Verdi	2.16	5.61	41.33	0.50	0.50	0.53
	PSI	2.05	3.81	46.01	0.44	0.52	0.56
	PRI-PSDI-PLI	0.77	7.22	55.42	0.47	0.64	0.69
	DC	0.33	21.44	39.77	0.33	0.62	0.67
	MSI	0.14	0.45	75.94	0.21	0.77	0.84

In conclusion, with this chapter I have created tools to blend different vote flow matrices into an enhanced one, with better explicative powers and minimal error compared to the starting ones.

Two dimensions were blended - the positional competition (in the expected and mode values) and the electoral flows - and their relation tested empirically and significatively.

A concrete application was presented, estimating the expected and mode values for each party for the elections of 2018 and 1992. These estimates were also used in my analyses in chapter six.

More in general, this methodology is extremely useful to estimate the ideological positioning for each party in a given election, starting only from the positioning and the electoral flows from the previous or the next election.

PART III NORMATIVE ANALYSIS

Chapter 8

Building electoral systems

8.1. Introduction

This chapter aims to provide new tools for electoral system design, offering an unprecedented, more equal and respectful optimization of external and internal costs, introducing new variables in consideration of the much complex reality in which they are applied.

New differential calculus optimizations are then solved, aiming to building more equal electoral systems. I considered conflict channels inside the assembly as well as related to the number of electors that each MP must represent (as done by Taagepera) and I also included variables M and N. I also considered two other conflicting cleavages: 1) the respect of the electors' preferences - the representation - and 2) the cabinet stability which, conversely, implies a dis-representation.

In detail, I unparallelly cast into system the relations, and then the optimization, of extended equations using political and institutional variables P, S, M, N. This optimization will help determine the general features of an optimal electoral system and its institutional shapes (M and S), for a given country, with a particular population and party system (N). The resulting optimal value S^* will be more accurate than in previous literature, and M^* unprecedented. A star * after a variable's name indicates the optimal value of the same variable.

After these passages, I isolate the share of total dis-representation (D_2) attributable to the electoral system (D_E) and build a new conflict equation which considers these components as well as the dis-representation generated proportionally to the cabinet stability. In this way, D_2 and D_E will also be optimized.

In more detail, this chapter calculates the optimal value of S more accurately than the simple relation P=S^3 established by Taagepera *(2007b, p. 199)*, and unparallelly it also formalizes the optimal value of M. Moreover, the second conflict equation introduces the original index D_E, representing the dis-representation attributable to an electoral law and to the institutional shapes M and S (independent from N).

In order to minimize conflict channels – reported in the four pillars -, I propose two different methodologies, depending on the optimal variable I want to obtain. In the first methodology, I obtain the optimal values for S and M using differential calculus that minimizes the conflict channels produced by P, S and M, but also taking into consideration N merged with S, net of some parameters - as never done before -. In the second methodology, I obtain the optimal values for D_2 and D_E using maximization calculus between the cabinet duration and a new comprehensive representation index RE, which is equal to the complement of Gallagher's index of dis-representation $D_2 = [0.5 \sum_{i=1}^{N_0} ⟦(s_i - v_i)^2⟧]^{0.5}$ *(1991)*, such that $RE = 1 - D_2$, which includes D_E and is standardized for the cabinet life (C) maximum duration.

The only genuinely exogenous variable is P, whereas the other variables (M, S, D_2 and D_E) are endogenous; N is to be considered a hybrid variable, since, in line with the above-mentioned time-series approach, it is assumed to be stable, on average, from one election to the other (as previously tested on a worldwide basis). All conflict channel formulas are derived for endogenous variables; the formulas are not derived for P or N, given their exogenous or hybrid nature. However, theoretically, it is possible to find the optimal value of N at the end of all optimization passages.

As for the first methodology, in supporting the introduction of N into the grafting of the assembly size (S) optimization, I start considering that parties play as an essential role in parliamentary dynamics as outside, representing the electors of a district; in particular, parties have the function of grouping people with a relatively common political view, and represent the base of cleavage formation (Heath, 2005) (Lipset and Rokkan, 1967) (Rae - Taylor, 1970) and of politics issues (Taagepera and Grofman, 1985), reducing de facto these said issues, and consequentially the conflict channels, versus simply considering S.

This reasoning is also applicable outside the chamber of representatives: candidates and their leadership for sure influence vote behavior; this behavior is also affected by common opinions among representatives on a program or ideology towards ideals and identity (Budge - Robertson - Hearl, 1987) (De Sio, 2011, p. 57-8), hence the parties. Notwithstanding knowing that parties can converge (more or less all representatives) on valence issues (Stokes, Spatial Models of Party Competition, 1963) (Lewin, 1935). This implies that not all MPs effectively represent an active part in the determination of a conflict channel, therefore the Effective S will be defined between S and N, hence requiring the application of the weighted geometric mean of these.

Operatively, this means setting the following expression: $N^{(e)} * S^{(1-e)}$, with exponents e [0,1] and *1-e* as weights, calculated in function of their respective average values N and S. In detail, the parameter e is a theoretical parameter calculable by substituting the empirical values P, S, N, in the formula of S^*, for as many countries as possible, considering the most recent values available, and finally applying the geometric mean to the numeric values of *e* obtained for all countries. Following the same methodology, it is possible to obtain M^* by substitution of S^*.

Moving to the second methodology of maximization of the cabinet duration and R, I start by noting the relation between D_2 and the institutional variables M, S, and N such that $D_2 = \frac{0.5}{N^2} = \frac{0.5}{\sqrt[3]{MS}}$ (Shugart - Taagepera, 2017, p. 145-6). I can then substitute $N_s{}^2$ in the function $C = \frac{k}{N_s{}^2}$ with the geometric mean between: 1) $N_s{}^2$; 2) $\sqrt[3]{M^*S^*}$; 3) $1/2D_E$ (by definition). In such a way, I have nested M^* and S^* inside the previous calculations. I proceed to calculate the RE index, simply applying the basics of the probability for independent events, standardized for C. Finally, I obtain $D_2{}^*$ probabilistically from $D_E{}^*$.

In conclusion, an innovative finding presented in this chapter is that the higher N_s and/or M*S* product, the more $D_E{}^*$ tends to 0; conversely, the lower N_s and/or M*S* product – both tending to 1 -, the more $D_E{}^*$ tends to 1, describing the correlations between N_s and $D_E{}^*$ and between M*S* and $D_E{}^*$ as branches of the hyperbola.

8.2. The conflict channels

Current literature on optimisation in politics mainly focuses on assembly size S, obtained from the minimization of the conflict channels c (Taagepera R. , 2007b, p. 189-191), in its two components a) and b):

a) how many citizens each MP must represent ($\frac{2P}{S}$), in a relation of reciprocal interaction (hence why the ratio P/S is multiplied by 2);

c) how members of the assembly interact with each other ($\frac{S^2}{2}$). This is derived from the formula $\frac{\boldsymbol{n(n-1)}}{\boldsymbol{2}}$ which can be rounded to $\frac{n^2}{2}$ for high values of n (Taagepera R. , 2008a, p. 140; 2007b, p. 198-99).

This is reflected in the formula for conflict channels equal to $c = \frac{2P}{S} + \frac{S^2}{2}$.

I then apply the differential equation derived by S equal to $\frac{dc}{dS}\left(\frac{2P}{S} + \frac{S^2}{2}\right) = 0$ if P >1000 (Taagepera R. , 2007b, p. 198-9) resulting in the optimal S, that I call S^*, equal to $S^* = \sqrt[3]{2P}$.[84] Taagepera continues imposing the assumption that the active population corresponds more or less to P/2, hence re-formulating the previous equation into $S^* \cong \sqrt[3]{P}$ (Cfr. (Taagepera R. , 2007b, p. 199)).

I am interested in adding some other political variables which I believe have an impact on the optimisation of the conflict channels mentioned above.

Electors in the district will often refer to parties more than to their representatives, which they may not know in person. Therefore, as the optimization of S must consider the minimization of conflict channels of the representatives inside the assembly and the district, I will also consider parties both inside and outside the assembly.

For all these reasons, I can picture a mesh of the parties' components given by the effective number of parties in terms of seats[85] $N_{2,s}$ - simply displayed as N – and S, in substitution of the simple S in the previous differential calculus $\frac{dc}{dS}\left(\frac{2P}{S} + \frac{S^2}{2}\right) = 0$, thus obtaining $\frac{dc}{dS}\left(\frac{2P}{(SN)^{0.5}} + \frac{SN}{2}\right) = 0$. In absence of other information, I have elevated the product between S and N to the power of 0.5, representing the geometric mean between the minimum (N) and the maximum (S) of the range (Cf. Taagepera (2008a, p. 120-129)).

Finally, optimizing only for these variables, I obtain $S^* = \frac{\sqrt[3]{4P^2}}{N}$; for all passages see appendix 8.1. Nevertheless, the previous formula cannot be applied as is, as it would produce inconsistent results, due to the weights of N and S not always fixed at 0.5, as introduced in section 8.1 and mentioned above. Therefore, the weights of N and S will need to be parametrized. Moreover, it could be possible to obtain further information regarding the political system, by quantifying the impact of dis-representation.

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[84] Cfr. (Taagepera R. , 2015, p. 170-2).

[85] Which is known to be related to the political issues (Taagepera and Grofman, 1985).

I proceed to formalize these improvements, starting with this latter point of quantifying the impact of dis-representation as the next step.

8.3. The assembly size

I must add some other political variables, such as the dis-representation (D_2): an increase of D_2 would mean a lack of "sincere" expression of electors' preferences, implying that the SN product would be ineffective, inevitably producing a parliament with minor conflict channels, because the representatives will be more homogeneous.

The limit point of maximum homogeneity of representatives is $D_2 = 1$, in which the strongest electoral law would fill all seats with one party, homologating the assembly, producing an effective SN product which would tend to S.

Considering that a democratic regime could theoretically tend to $D_2 = 0.5$, where it is possible to imagine other dynamics in the formation of blocked lists which could anyway produce an SN product equal to 1. This would mean that something in the electoral system and/or parties' strategies has intervened to nullify the impact of the effective number of parties and the members of the assembly, leading de facto to a dictatorship.

The other limit case is for the SN product to be equal to itself, when D_2 is equal to 0, meaning that the political system is so representative that no effect on the conflict channels happened.

Thus, I can write: $\frac{dc}{dS}\left(\frac{2P}{(SN)^{0.5(1-D_2)}} + \left(\frac{SN}{2}\right)^{(1-D_2)} - D_2\right) = 0$. A further simplification can be performed substituting the variable d, such that $d = 1 - D_2$.[86] At the end of all differential calculus I obtain $S^* = \frac{P^{\frac{2}{3d}}}{N}$; for all passages see appendix 8.2. Nevertheless, this formula is inapplicable as is, as it would produce inconsistent results, due to the weights of N and S not always fixed at 0.5, as said before.

Is in fact possible to obtain further information regarding the political system, quantifying the impact of N, introducing the parameter "e" defined in the interval [0,1], which will be inversely proportional to the

[86] Several attempts have been made to linearize the impact of parameters d and e (soon introduced); nevertheless, the result would be inconsistent with the logical constraints and the existence domain.

impact of S (on the SN product), which has a parameter of (1-e). The final product will therefore be equal to: $N^{(e)} * S^{(1-e)}$.

The parameter e is a theoretical parameter calculable by substituting the empirical values P, S, N, in the formula of S^{*}, for as many countries as possible, considering the most recent values available, and finally applying the geometric mean to the numeric values of e obtained for all countries. This parameter, empirically evaluated, is fundamental to determining the final results of the proposed formulations which will follow; otherwise, if set arbitrarily, it will yield inconsistent final results. A further corrective is required such that the SN product $(N^{(e)}S^{(1-e)})$ must fulfil the following conditions: 1) for e=0 it must be equal to S, and for e=1 it must be equal to N, 2) for e=0.5 the result must be equal to $SN^{0.5}$.

I therefore propose the following exponent: $1-(-4e(1-e)+1)$, as shown:

$$(N^{(e)}S^{(1-e)})^{1-(-4e(1-e)+1)}$$

Nevertheless, this is an unfortunate formulation because an exponential of exponential appears, whose derivative produces a very complex and hard-to-manage expression, and as such it must be simplified. I know that, for differential calculus the assumption is valid that the simplest the conflict function to optimize, the simplest the result, particularly with regards to the exponent. Hence, applying the criterium of the geometric mean, the final solution will be simply equal to the exponent 2.

In fact, analysing the anchor points by e: for e=0 the SN product is equal to S, for e=1 the result is N, and for the mid-point e=0.5 the result would be $(SN)^{0.5}$ (exactly equal to the geometric mean between S and N) halving the whole SN's exponent. Therefore, the final exponential corrective applied, equal to 2, allows to satisfy the above-mentioned conditions rebalancing the SN's member.
Summarizing:

$$\frac{dc}{dS}\left[\frac{2P}{\left(N^{(e)}*S^{(1-e)}\right)^{(d)}}+\left(\frac{S^{(2-2*e)}*N^{(2*e)}}{2}\right)^{(d)}\right]=0,\ being\ d,P,S,N,S,e\geq 0$$ [87]

[87] The equation obtained from this passage is equal to: $2^{(1-d)}d(e-1)N^{(-de)}S^{(d(e-1)-1)}\left(2^{d}P-N^{(3de)}S^{(-3d(e-1))}\right)=0$, which I then solved for S, leading to the next step.

$$S^* = \left(\frac{2^{(-d)}N^{(3de)}}{P}\right)^{\frac{1}{3d(e-1)}} \Longrightarrow S^* = \frac{N^{\frac{e}{e-1}}}{2^{\frac{1}{3(e-1)}}P^{\left(\frac{N^2}{3(e-1)(N^2-0.5)}\right)}}$$

For all passages see appendix 8.3. It is important to highlight that the parameter e is a theoretical parameter calculable by substituting the empirical values P, S, N, in the formula of S^*, for as many countries as possible, considering the most recent values available, and finally applying the geometric mean to the numeric values of e obtained for all countries.

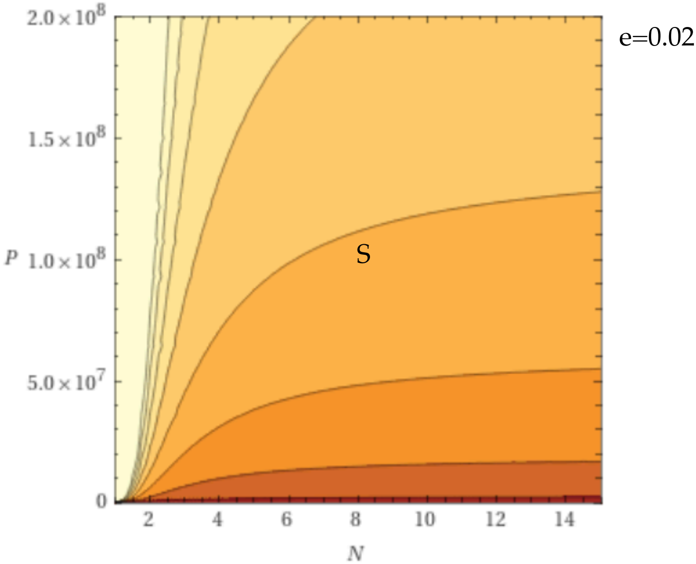
The last formulation can be further simplified by factorizing e into numeric constants f and a as follows: $f = \frac{1}{3(e-1)}$ and $a = \frac{-e}{e-1}$. The fully simplified formula becomes:

$$S^* = \frac{1}{2^f N^a P^{\left(\frac{fN^2}{(N^2-0.5)}\right)}}$$ [88]

The contour plots below explore the overall impact of the variables for parameter e equal to 0.02 (Figure 49) and 0.1 (Figure 50).

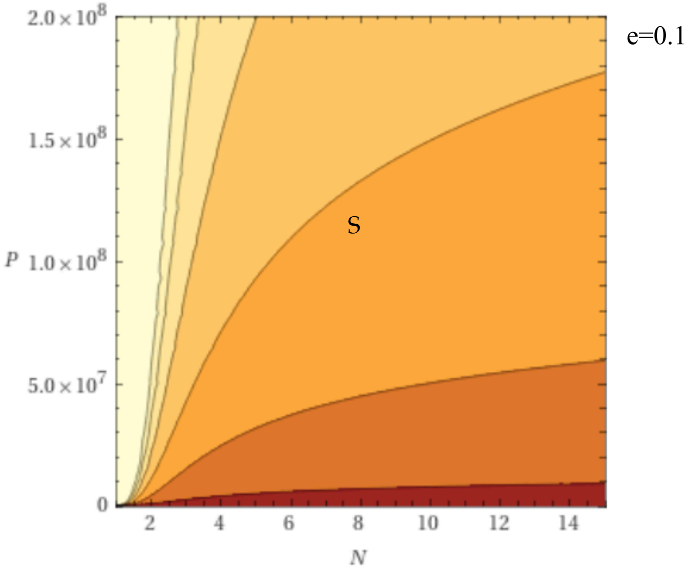
[88] Valid for P equal to active population and >1000.

Figure 49 Contour plot graph of S, in function of P and N; imposing the parameter e=0.02.



In Figure 49 parameter e is imposed equal to 0.02. For example, substituting N=3 and P=1.25*10^8, the seats are 568.

Figure 50 Contour plot graph of S, in function of P and N; imposing the parameter e=0.1.



In Figure 50 the parameter e is imposed equal to 0.1. When compared to previous Figure 49, both graphs show that the lower N, the higher P, the higher S; conversely, the higher N, the lower P, the more S tends to 0. It is also possible to observe that ceteris paribus an increase of the parameter e goes to produce a reduction of S.

These results show that the higher e, the higher S^*; this means that the higher the weight of N on S, the more the parliament's homologation: representatives are not independent from their belonging party and they are flattened on the political lines of their respective parties. The higher the e, the more an increase of the S is opportune to rebalance this flattening, hoping that more representatives could have some independence to propose more policies and therefore more issues in parliament. The higher N, the lower S because there are enough policies and issues brought by many effective numbers of parties in parliament, and many MPs would be plethoric.

Lastly, I also confirm the original relation that sees the cube root of the population variable as proxy of S: in fact, the more numerous the population, the more numerous representation it requires, following a logarithmic rhythm.

Given these considerations, I can confirm that the results obtained for S^* through differential calculus are also logically reasonable.

8.4. The magnitude of the district (deputies by district)

The optimization of M is also possible using the mother relation ${N_2}^3 = {N_0}^2 = (M * S)^{\frac{1}{2}}$ [89] . Considering $N_2 = (M * S)^{\frac{1}{6}}$, I simply substitute N with $(M * S)^{\frac{1}{6}}$ in the previous differential equation, and derive it by M (instead of S), thus obtaining the optimal M – which will be M^*-. In addition, I use the already found optimal value of S – S^*- and I have:

$$\frac{dc}{dM}\left[\frac{2P}{(S^{*(1-e)}(MS^*)^{e/6})^d} + \left(\frac{S^{*(2-2e)}(MS^*)^{e/3}}{2}\right)^d\right] = 0, \mathit{being}\ d, P, S, N, S, e \geq 0$$ [90]

$$M^* = 2^{(2/e)} S^{*(5-6/e)} P^{(2/(de))}$$

For all passages see appendix 8.4. From here, the optimisation calculations must substitute and use the equation $D_2 = \frac{0.5}{N^2}$ (Shugart - Taagepera, 2017, p. 146-7), therefore $d = 1 - D_2 = 1 - 0.5/N^2$, I obtain:

$$M^* = 2^{\frac{2}{e}} S^{*\,5-\frac{6}{e}} P^{\overline{\frac{2}{(1-0.5/N^2)e}}}$$

As done before, this last formulation can be further simplified by factorizing e into numeric constants b and c as follows: $b = \frac{2}{e}$ and $c = -(5 - \frac{6}{e})$. The fully simplified formula becomes:

$$M^* = \frac{2^b P^{\overline{\frac{b}{(1-0.5/N^2)}}}}{S^{*\,c}}$$

Following previous considerations, the more numerous the population, the more numerous representation it requires, the higher M must be because the higher the probability of dis-representing many people in a

[89] For an in-depth treatise see the introduction and (Taagepera R. , 2007b, p. 97; 154-156; 226).

[90] The equation obtained from this passage will be equal to: $1/3\mathrm{DeM}^{(-(\mathrm{De})/6-1)}\mathrm{S}^{(1/6\mathrm{D}(5\mathrm{e}-6))}\left(0.5^{\mathrm{D}}\mathrm{M}^{((\mathrm{De})/2)}\mathrm{S}^{(\mathrm{D}(3-(5\mathrm{e})/2))} - \mathrm{P}\right) = 0$ which, after being solved for S, will take to the next step.

small district (as in FPTP) (Fisichella, 2009, p. 283) (Taagepera R. , 2007b, p. 206-211).

As for the relation between N and M, the more effective number of parties, the lower M, allowing the party system to rebalance the fragmentation dynamics; conversely, the lower N, the higher M, since like in a monetary policy, this could be seen as an injection of liquidity, hence a stimulus to the party system to generate more politics and issues (through new parties).

In politics, this is enunciated by Colomer in the micromega rule «The small prefer the large, and the large prefer the small» (2004, p. 3) with reference to parties on one side and S and M on the other side of the enunciation. Duverger capitalizes on this rule in his "first law", which is applied to the parties and M (and not S), such that: a majoritarian electoral system is going to produce a bi-polar competition (1951; 1954, pp. pp. 247, 269; 1955, p. p. 113).

Finally, in relation to the impact of S^* on M^*, knowing that the constant $c > 1$,[91] the higher S, the lower M, following the exact same dynamic as above: the more plethoric the assembly size S, the narrower the district magnitude M must be to contrast this. In conclusion, the results obtained through the differential calculus seem to be logically reasonable also for M^*.

8.5. The dis-representation due to the electoral systems

I now introduce another measure to be optimized - D_E -, which consists of the portion of dis-representation D_2 produced by electoral laws, logically defined in the interval [0,0.5] like D_2. This index will measure the aggregated effects of: the introduction of the threshold of representation, majority premiums, and the intrinsic mechanics of the logics of allotment of seats from votes.

I can simply consider the dis-representation index of Gallagher D_2,[92] as formed jointly by three independent components: 1) D_E, which is the dis-representation produced by the electoral system; 2) $\frac{0.5}{N^2}$, which is the dis-representation produced by the party system, as introduced previously;

[91] Since for $e = 0 \Rightarrow c = \infty$, and for $e = 1 \Rightarrow c = 1$.

[92] The Gallagher index is: $D_2 = \left[0.5 \sum_{i=1}^{N_0} (s_i - v_i)^2\right]^{0.5}$ (Gallagher, 1991).

3) $\frac{0.5}{\sqrt[3]{M^{*}S^{*}}}$, which is the dis-representation produced by the institutional variables M and S (Shugart - Taagepera, 2017, p. 144-6).

As these three components (events) are independent, and in absence of a statistical formula for 3 factors, I can hypothesize that the interaction produced in order to subtract their summations (necessary to aggregate independent probabilities) could be the geometric mean of the components. In fact, if one component was 0, and the other two were equal to 0.5 and 0.5, D_2 would be equal to 1, which would be out of the possible empirical limit [0,0.5]; instead, with the geometric mean – in case one component was 0 – ordering the elements (in this case, the components) by their size such that $a_1 < a_2 < a_3$, I can substitute a_1 in function of a_2 and a_3 using the formula $a_1 = \frac{{a_2}^2}{a_3}$ (Taagepera R. , 2015, p. 212-3), and solving for the previous scenario, I would obtain $D_2 = 0.5$, which is within the empirical limit [0,0.5], as logically required.
Hence the formula for D_2 composed by the three factors is:

$$D_2 = D_E + \frac{0.5}{N^2} + \frac{0.5}{\sqrt[3]{MS}} - 2\sqrt[3]{\frac{0.25D_E}{\sqrt[3]{MS}}}$$

It is possible to logically validate this formula by studying its anchor points: 1) for all factors equal to 0, $D_2 = 0$; 2) for all factors equal to 0.5, $D_2 = 0.5$.

Looking at the latter anchor point, 0.5 is fixed as the empirical limit for electoral democracies: in fact, a dis-representation higher than 0.5 would mean that more than 50% of votes would be mis-represented, thus also breaking the basic rules of representation. Simplifying, if a 2 party system registered a $D_2 > 0.5$, we would be in presence of a dis-representative scenario where, at minimum: party A which has 0 votes takes at least 50% of seats, and conversely party B with 100% of votes would surely take less than 50% of seats, then completely meaningless for a democratic legitimation of a parliament.

8.6. The overall dis-representation

The question is now how to optimize D_E. I consider the idea that two opposite interests exist: on one hand the Cabinet duration, on the other the representativeness. These work in opposite ways because an electoral law can increase the Cabinet duration, minimizing the party fragmentation (Taylor H.M.- Herman V.M., (March)1971), but in doing so the representation would deteriorate.

Therefore, I must maximize: 1) the cabinet duration; 2) the degree of representation produced by the electoral law – inversely proportional to D_E - and which derives from the institutional shape – also inversely proportional to M and S.

With regards to the operationalization of the cabinet duration, I can consider the equation $C = \frac{k}{\mathrm{N}_s{}^2}$ (Taagepera - Shugart R.-M. , 1989, p. 99-101), where k is a constant and N_s is the effective number of parties calculated on seats. Taagepera's most recent empirical evaluation of k was performed on 26 democracies, obtaining k=42 (Taagepera R. a., 2007, p. 168-9).

With regards to the degree of representation, I can consider the previous D_2 equality, with some adjustments. First of all, I define the representation index – RE - logically defined in [0,1], as the complement of D_2 such that: $RE = 1 - D_2$; I then need to isolate the portion of dis-representation D_2 produced by the electoral law and derived from the institutional shape: using the classical probability of independent events (by two factors)[93] I obtain: $RE = 1 - \left(D_E + \frac{0.5}{\sqrt[3]{M^*S^*}} - \frac{D_E}{\sqrt[3]{M^*S^*}}\right)$.

I must fulfil two final conditions: 1) capitalize the impact on representation of the effective number of parties, calculated on votes N_v; 2) RE could empirically assume the value of 0. For $N_v = 1$, the impact on representation of D_E and M^*S^* would tend to 0 because electors have decided – before the electoral and institutional shape – that they need only one effective party, therefore RE will tend to 1 (this scenario is only a logical anchor point, nevertheless unrealistic among democracies); when N_v tends to infinitive, the disproportional impact of D_E and M^*S^* will be outclassed, therefore producing an overall RE tending to 0.

The simplest formulation that fulfils the previous conditions could be $RE = \frac{1-\left(1-\frac{1}{N_v}\right)\left(2D_E+\frac{1}{\sqrt[3]{M^*S^*}}-\frac{2D_E}{\sqrt[3]{M^*S^*}}\right)}{N_v}$, in which each of the three components of D_2 has been multiplied by two so to fulfil the condition of a domain included in the interval [0,1].

[93] For a statistical explanation see (Espa - Micciolo, 2008, p. 51).

Finally, to obtain a maximisation of equal entities, I must define the second term of the differential calculus in the same interval's existence: being $\frac{k}{{\mathrm{N}_s}^2}$ defined $\forall\ k > 0$; C ∈ [0, k] and RE ∈ [0,1], it will be sufficient to multiply RE for k. Therefore, I obtain the conflict function - Co - equal to:

$$Co = \frac{k}{{\mathrm{N}_s}^2} + kRE \Longrightarrow$$

$$Co = \frac{k}{{\mathrm{N}_s}^2} + k\left(\frac{1 - \left(1 - \frac{1}{N_v}\right)\left(2D_E + \frac{1}{\sqrt[3]{M^*S^*}} - \frac{2D_E}{\sqrt[3]{M^*S^*}}\right)}{N_v}\right)$$

It is now sufficient to derive the formula above by D_E and set it equal to 0 to find the maximum, as follows:

$$\frac{dc}{dD_E}\left[\frac{k}{{\mathrm{N}_s}^2} + k\left(\frac{1 - \left(1 - \frac{1}{N_v}\right)\left(2D_E + \frac{1}{\sqrt[3]{M^*S^*}} - \frac{2D_E}{\sqrt[3]{M^*S^*}}\right)}{N_v}\right)\right] = 0$$

A consequence of this differential calculus is that the argument $-D_E$ - will disappear, because it does not occur in both terms of the differential calculus; for all passages see appendix 8.5. However, we must not despair as MS, N and D_E occur to determine D_2 through the previous equivalences of $N^2 = \sqrt[3]{\mathrm{MS}} = \frac{1}{2D_2}$; knowing that D_E and D_2 exist in the same domain, and that D_E is an independent probabilistic factor of D_2, I can approximate $N^2 = \sqrt[3]{\mathrm{MS}} = \frac{1}{2D_2} \approx \frac{1}{2D_E}$.

Applying the geometric mean (because all factors are integer and >1) and using the optimized M* and S*, I obtain that the cabinet duration is $C = \frac{k}{{\mathrm{N}_s}^2} \cong \frac{k}{\sqrt[3]{\frac{{\mathrm{N}_s}^2\sqrt[3]{M^*S^*}}{2D_E}}}$ .Thus the differential formula becomes:

$$\frac{dc}{dD_E}\left[\frac{k}{\sqrt[3]{\frac{{\mathrm{N}_s}^2\sqrt[3]{M^*S^*}}{2D_E}}} + k\left(\frac{1 - \left(1 - \frac{1}{N_v}\right)\left(2D_E + \frac{1}{\sqrt[3]{M^*S^*}} - \frac{2D_E}{\sqrt[3]{M^*S^*}}\right)}{N_v}\right)\right] = 0 \Longrightarrow$$

$$D_E{}^* = \frac{N_v{}^2 \sqrt[3]{M^* S^*}}{6\sqrt{3}(N_v - 1)\left(\sqrt[3]{M^* S^*} - 1\right)\sqrt{\frac{(N_v - 1)\mathrm{N}_s{}^2\left(\sqrt[3]{M^* S^*} - 1\right)}{N_v{}^2}}}$$

This is the comprehensive formula to calculate $D_E{}^*$, which returns the most accurate results; for all passages see appendix 8.6. However, in order to proceed with further analysis, I propose to simplify this formula, rewriting the second term of the differential equation in a way that still respects the inverse correlation between RE and N, but is less prescriptive over the noteworthy points: in my proposal, RE tends to 1 for smaller and smaller values of N_v tending to 1 – without becoming equal to 1 for N_v =1 -, and RE tends to zero for higher and higher values of $N_v \to \infty$ - without becoming equal to zero for $N_v \to \infty$ -.

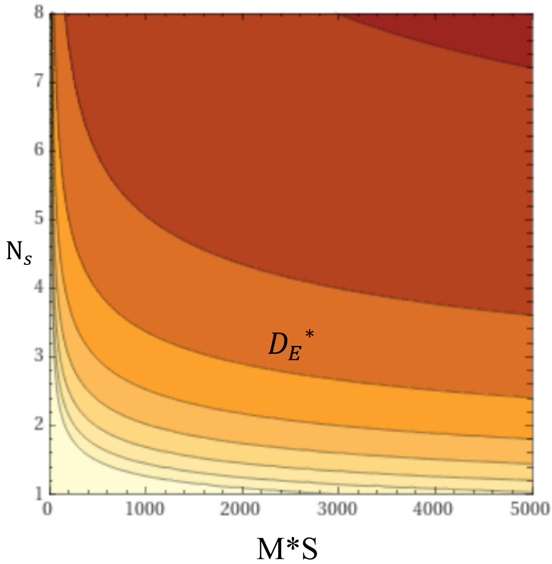
I also consider the dis-representation generated by the electoral systems (by M, S and D_E) net of that generated by the effective number of parties (calculated on votes) resulting in $\frac{0.5}{N_v{}^2}$, plus a corrective to guarantee the domain [0,1]. The simplified formula is:

$$\frac{dc}{dD_E}\left[\frac{k}{\sqrt[3]{\frac{\mathrm{N}_s{}^2 \sqrt[3]{M^* S^*}}{2D_E}}} + k\left(0.5 + \frac{0.5}{N_v{}^2} - \left(D_E + \frac{0.5}{\sqrt[3]{M^* S^*}} - \frac{D_E}{\sqrt[3]{M^* S^*}}\right)\right)\right] = 0 \Rightarrow$$

$$D_E{}^* = \frac{\sqrt{\frac{2}{3}}\sqrt[3]{M^* S^*}}{3\left(\sqrt[3]{M^* S^*} - 1\right)\sqrt{\mathrm{N}_s{}^2\left(\sqrt[3]{M^* S^*} - 1\right)}}$$

For all passages see appendix 8.7. Figure 51 below shows the contour plot of the overall impact of variables M, S and N_s on $D_E{}^*$:

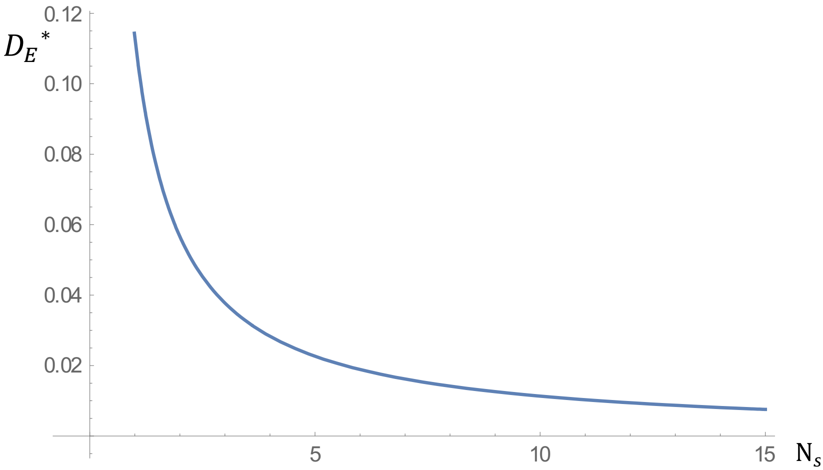
*Figure 51 Contour plot of the dis-representation produced by the electoral system $D_E{}^*$, in function of $N_s$and M*S* product.*



The higher N_s and/or M*S* product, the more $D_E{}^*$ tends to 0; conversely, the lower N_s and/or M*S* product, the more $D_E{}^*$ tends to 1.

Figure 52 below shows the inverseley proportional relation between optimal disrepresentation $D_E{}^*$ and N_s.

*Figure 52 The 2D graph of $D_E{}^*$ on N_s; imposing M*S*= 600.*



Assigning a numerical value to M*S*, in this case 600, I can see how $D_E{}^*$ is inversely proportional – more than proportional - to N_s, tending to 0 for $\mathrm{N}_s \to \infty$ whereas, for N_s tending to 1, $D_E{}^*$ would tend to ∞ because the function is a branch of the hyperbola. Nevertheless, recalling the previously stated optimization condition P>1000, this goes to reduce the value of $D_E{}^*$ to the limit of 1 for N_s tending to 1, and/or for values of M*S* tending to 1, with reference to the insights provided by figure 51.
A noteworthy point could be N_s=2, which corresponds to a pure bi-party competition; in this point, $D_E{}^*$ is about 0.055.

Going back to the first definition of D_2, I can substitute all optimized elements such that I obtain:

$$D_2{}^* = D_E{}^* + \frac{0.5}{N^2} + \frac{0.5}{\sqrt[3]{M^* S^*}} - 2\sqrt[3]{\frac{0.25 D_E{}^*}{N^2 \sqrt[3]{M^* S^*}}}$$

In conclusion, with the assumption of P, e, N_v (and N_s if I wanted to be more precise) as endogenous variables, and obtaining M^*, S^*, $D_2{}^*$ *and* $D_E{}^*$, I have all the tools necessary to project an electoral system. Using the existing tools presented in the available literature (Taagepera R. , 2007b); (Shugart - Taagepera, 2017) and some of the formulas that I obtain in this dissertation, it is possible to evaluate the impact on the desired electoral system of each single tool such as threshold, majority premia, PR rules, and also to recalculate their impact on the effective number of parties.

These models offer some directional and quantitative relations between institutional and political variables. In order to determine the exact value of S and M it is necessary to calculate the value of e, as shown above. Furthermore, in order to validate the optimized values for S, M and D_2 it will be necessary to empirically test their correlation with their respective actual values, using again the dataset of Struthers - Li - Shugart (2018), expanded to include the data on P (manually).

Chapter 9

The electoral general equilibrium of the electoral strategies

9.1 Introduction

This chapter wants to determine an equilibrium between parties' and voters' "electoral utility". This kind of equilibrium represents a pivotal point in building in details the electoral systems. In fact, an electoral system generates some dis-representation, which will benefit some groups and disadvantage others. The "electoral utility" is the quantity of dis-representation which benefits a group of parties and voters in the system, producing disutility for the others. With greater dis-representation, major parties tend to maximize their utility by approving electoral laws capable of maximizing their seats; conversely, some voters see a decline of their utility as they are prompted to express an insincere vote when they would have not voted such parties as their first preference.

The previous chapter has undoubtedly laid the foundations of an optimal electoral system in general terms, through an optimization of institutional and dis-representation variables; this chapter complements such view by also taking into consideration some features of electoral laws, such as the FPTP system, thresholds and majority premiums (MJPs). I leverage LQMs, in much simplified forms, by connecting them in a primary game theory approach and find a theoretical equilibrium, driven by the political theory axiom of different social sciences interests (of electors and parties) founded on the "Maximin" Rawlsian theory (1971).

In this chapter I follow a normative approach to further specify the role of electoral system correctives in function of strategic vote and the more theoretical Rawlsian criterion mentioned above.

As with previous considerations, the law of minority attrition provides an excellent starting point due to its very versatile applications, from proportionality to FPTP and all shades in between. In this case, I have reshaped its bottom left corner to take into account the impact of the block threshold in that part: the threshold, if present, impacts the number of seats allocated in correspondence of the votes received up to its limit, determining a snapped curve in its correspondence and zero seats before. Specularly, in presence of a majority premium, the curve of the law of minority attrition can be reshaped in the top right corner, to reflect a jump because a majority premium creates a net upgrade of seats. This approach allows multiple scenarios possible thanks to the consideration of the variating nature of institutional and/or political variables: as an example, in consideration of the fact that there exist infinite combinations of para-institutional variables T and MJP, these could be reduced by substitution of the optimized variables S and M from chapter 8, to re-determine N and G.

Finally, using the optimal values found before – or not - and some other tools already available (R. Taagepera 2007b) (Shugart - Taagepera 2017), this chapter provides: 1) a set of tools to determine party and elector equilibrium strategies; 2) a simulation and design of electoral systems, evaluating how seats' allotment varies with a change in electoral rules and then strategic votes in relation to threshold, majority premiums, but also any other corrective and proportional rule characteristic; 3) electoral projections through survey data, creating forecasts to understand, for example, if the mix of a specific electoral system with a specific political system goes to generate a majority in the assembly on not.

9.2 The puzzle

The concepts formalized so far are not just useful for the description of a party system, but also for the detection of strategic vote, which I define as follows:

Strategic vote is the vote given by electors to a party which does not coincide with their first preference in their conceptual order of party preference, because of the major probability of success that they attribute to the voted party. The probability of success of a party could be conceptually represented – for now - by its seats on votes ratio.

Therefore, a strategic vote is logically expected when there is some dis-proportionality between votes and seats during the electoral process, from the expression of the vote to seats allocation.

Strategic vote cannot simply be linked to the second preference in an elector's conceptual order of party preference. In fact, the elector could change their preference several times before expressing their vote: particularly, it could happen that some i-th parties are preferred before that which is voted, being those without chances of victory, or simply because they are valuated with an inferior cost-benefit.

Strategic vote would not be triggered in an ideally perfectly proportional system, but rather in presence of some dis-representative elements, that would materialize even in proportional electoral formulas and more significantly in clearly majoritarian systems.

Practically, the measurement of the strategic vote is not a simple issue. Given the first law of Duverger (1951; 1954), which says that a plurality system (as well as majoritarian) is going to produce a bipolar competition, I would logically expect that the higher the votes received by a party, the more than proportional the seats given to it compared to the minor ones.

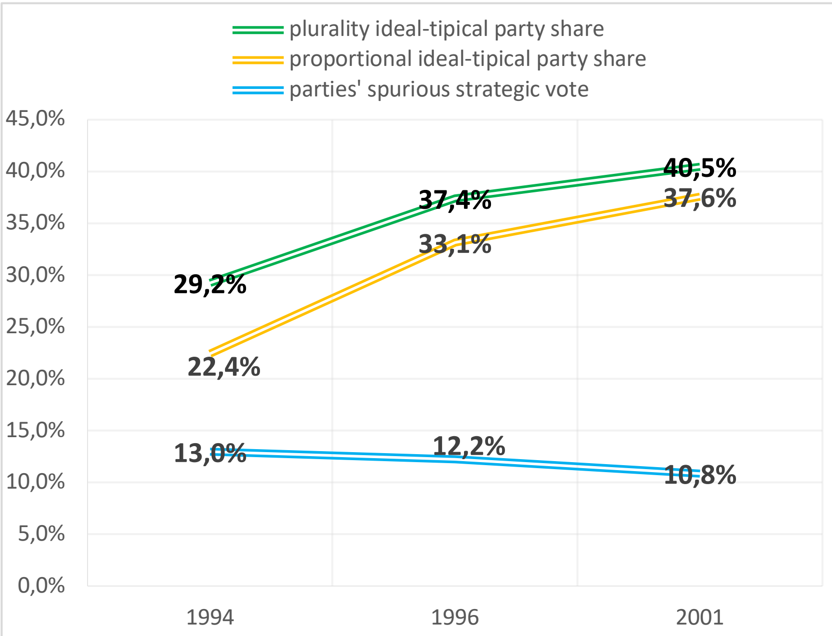
An example is the single-member district with a single winner (FPTP) as compared to the opposite proportional system, not based on the previous dynamics. In a single-member district electors are aware of the weight of their own vote (which will results in relatively more seats for the most voted parties respect the lesser), respecting the relation of seats allocated in function of votes received, traced by the law of minority attrition (Taagepera R. , 2007b, p. 208). The majoritarian dynamics will also emerge in proportional systems in presence of correctives such as the explicit block threshold, majority premium or any other dis-representative mix of allocations of votes in seats.

More widely, a party concentration dynamic is ontologically present in all electoral systems in form of the implicit block threshold T that is inversely proportional to the number of constituencies E and to the average size of the constituency M, in line with all vote dynamics mentioned so far; formally: $T = \frac{75\%}{(M+1)\sqrt{E}}$ (p. Id., 247).

The function implies that the maximum value of block threshold corresponds to the single-seat district (M=1) in which : $T = \frac{75\%}{2\sqrt{E}} = \frac{37{,}5\%}{\sqrt{E}}$; for all other values of M, T will be lower. Taagepera combined this formula with the nominal number of parties relation, obtaining: $N_0 = \sqrt{\frac{37{,}5\%}{T}}$ for the plurality system and $N_0 = \sqrt{\frac{75\%}{T}}$ for the proportional system (Id. (p. 249)), which confirms that, ceteris-paribus T, the plurality system has a lower number of parties.

I can only deduce that given a same party system with the same electors, the plurality electoral system will have an ideal-typical party dimension HH greater in the plurality system than in the proportional one. Given the possibility for voters to express a double preference (one for a party and one for a candidate for FPTP) during the Italian elections from 1994 to 2001, I have a potential opportunity to calculate the strategic vote, as graphically represented in Figure 53 below.

Figure 53 Strategic vote as difference of the ideal-typical party shares. Italian general elections 1994-2001.



The blue line indicates the spurious strategic vote, obtained simply summing the absolute values of the differences of each party share between the proportional and majoritarian electoral sheets, also including the strategic supply of the parties which tend to coalize in order to maximize their electoral profit.

It is the electoral engineering itself that determines the exigence of new measurement tools able to provide an exact measure and answer the following research questions:

1) calculate party utilities, strategies and then supply;

2) calculate the counter strategies of candidates and electors;

3) calculate the exact strategic vote:
   3.1) what is the exact dimension of a party share for which it is neither advantaged nor disadvantaged?
   3.2) what is the strategic vote for HH, minor parties and the dominant party?

4) calculate an equilibrium able to consider all these strategies in function of the institutional set up and the electoral law.

9.3 The generalized utility function of the party systems. Strategies of candidates and electors

To understand the existence of electoral strategies, I start from the FPTP system, as it is very dis-representative, and subsequently widen to a general function which can describe the proportional systems as well; I can then graft other dis-representative options on this generalized model.

I recall my re-elaboration of the law of minority attrition (Cfr. (Taagepera R. , 2007b, p. 207-209)), done in chapter two:

$$\mathrm{s} = \frac{\mathrm{v^n}}{\mathrm{v^n} + (\mathrm{N}-1)^{1-\mathrm{n}}(1-\mathrm{v})^{\mathrm{n}}}$$

As Taagepera states that the 2 opponents must obtain "exactly the same vote shares" (ibid. p. 209), it is the case to apply a corrective to $(N-1)$ which considers G, that measures exactly how party shares are distributed (independently from the share size):

$$\mathrm{s} = \frac{\mathrm{v^n}}{\mathrm{v^n} + [(\mathrm{N}-1)(1-G)]^{1-\mathrm{n}}(1-\mathrm{v})^{\mathrm{n}}}$$

I also recall the following formula for n obtained in chapter two:

$$n = \left[1 + \frac{2}{0{,}5\left(1 - \frac{P-S^3}{P*S^3}\right)} * \frac{(1-G_d)}{(\mathrm{N}_2-1)}\right]^{\frac{S-M}{SM-1}*\left[1-\frac{E}{\sqrt{P}}\right]}$$

This allows to supply tools for politicians to design strategies and counter strategies, considering the institutional and political variables and the shape of party share distribution, as well as tools for electors to estimate their voting strategies. The inflection point of the law of minority attrition will determine the party share for which it is neither advantaged nor disadvantaged, and therefore the strategic vote will be to the advantage of the parties on the right of such inflection point, and likewise to the disadvantage of those on the left.

In presence of a threshold T, existing in [0,1], three break functions are identified:

1) the effect is s=0 for any party share $v < T$.

2) Being xlim the limit of influence of the probability curve which describes the overtaking of the threshold, following the law of minority attrition within the domain of v [T, xlim], I must blend two curves: one is the law of minority attrition determined by the threshold with domain in [0, xlim] (posing T< xlim), and the other is the general law of minority attrition, before analyzed, for each party system [0,1].

Figure 54 Law of minority attrition seats (s (%)) on votes (v (%)), corrected by a threshold T.

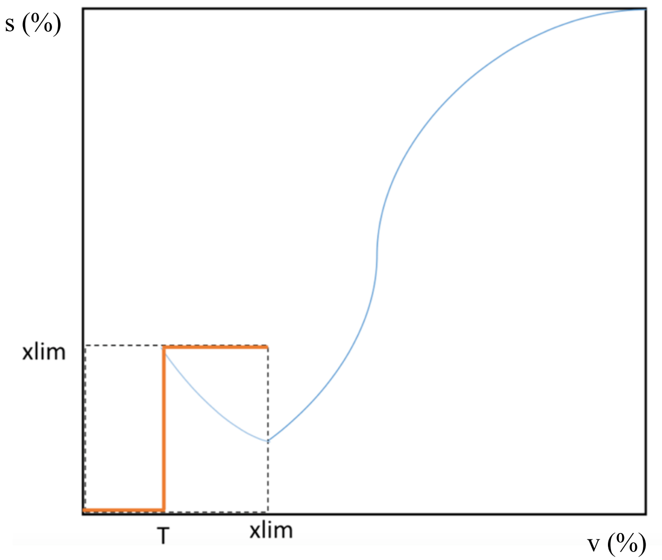


Figure 54 shows that the law of minority attrition, corrected by a threshold T, has a minimum and a maximum point, in addition to the anchor point (1,1). The maximum point in T is strategically explained by the minority parties' behavior as they will act to coalize with existing parties to try and achieve an aggregate share at least greater or equal to T, creating a sort of accumulation point exactly in T. The minimum point in xlim can be identified through a simple proportion:

$$[(N)(1-G)]^{-1}:1 = T:xlim \Rightarrow xlim = \frac{T}{[(N)(1-G)]^{-1}}$$

$[(\mathrm{N})(1-G)]^{-1}$ represents the index HH of the general law of minority attrition net of the corrective effective G and is here related to the general party system domain (which is 1), like T represents the HH of the "minor" law of minority attrition and is here related to its domain xlim.

I can now write the blended function defined in [T, xlim]:

$$\mathrm{s} = \frac{\mathrm{v}^{\mathrm{n}}}{\mathrm{v}^{\mathrm{n}} + [(\mathrm{N}-1)(1-G)]^{1-\mathrm{n}}(1-\mathrm{v})^{\mathrm{n}}} * \left[\frac{v-T}{xlim-T}\right] + xlim\left[\frac{xlim-v}{xlim}\right]$$

The weights of the laws of minority attrition are indicated in square brackets: the more v moves towards xlim, the more the weight of the general law of minority attrition moves from 0 and tends to 1, in an inversely proportional relation to the incidence of $xlim$ (function-value), with weight 1 when v=0 and 0 when v=xlim.

3) For v > xlim, the general law of minority attrition is applicable.

Moving now to measuring the impact of T on the number of parties, I can use the Taagepерian formula which links the number of parties with T and M. In its general form:

$$N_0 = \sqrt{\frac{\frac{75\%}{T}}{\left(1+\frac{1}{M}\right)}}$$

Taagepera states that for M=1 (plurality system) the formula reduces to: $N_0 = \sqrt{\frac{35\%}{T}}$, and for M "very large" (2007b, p. 248-9) I shall obtain $N_0 = \sqrt{\frac{75\%}{T}}$.

In line with my previous re-elaboration work, I can refine this formula introducing the exponent $\frac{S-M}{SM-1}$, which allows to define the range[94] of M from 1 to S: for M=1 the exponent is equal to 1, whereas for M=S this will be null.

$$N_0 = \sqrt{\frac{\frac{75\%}{T}}{2^{\frac{S-M}{SM-1}}}}, because\ of\ N^3 = {N_0}^2 \Rightarrow N = \sqrt[3]{\frac{\frac{75\%}{T}}{2^{\frac{S-M}{SM-1}}}}$$

At this point - in presence of a threshold - I must recalculate N with the above formula. Firstly, I must substitute: N calculated on the electoral parties, T=0, S known, and M to be determined - also because, as shown by Taagepera, the effective M could be different from the theoretical one (ibid. (ch.11)) -. Once M effective has been calculated, I substitute the value of T in the same last formula, to find the new N′ which will be used to recalculate the general law of minority attrition.

The research question is to determine the probability of the impact of the electoral law correctives, which corresponds to the area subtended by the law of minority attrition.

We can start by looking at the impact of the threshold on electoral supply. As all variables are now known, I can calculate the best electoral strategy in terms of party alliances on a territorial basis, represented by G, particularly calculating an equilibrium point G on the party supply side. This can be determined using integral calculus requiring integrating n itself, which would lead to complex formulations. Therefore, I propose to take a simplified approach by defining a model of correlation between n and G.

To define the correlation between n and G, I consider the direction of the Gini index, particularly whether party or parties, receiving a number of votes below the threshold, are excluded from seat allotment – on average – as the parties with votes above T benefit from this dis-proportionality: in this case, I hypothesize a decrease of G. Following my approach in previous models, I must estimate the overall size of the parties which could potentially be excluded. It is possible to use the previously re-adapted law of minority attrition as a cumulative probability distribution function of the overall system party shares s, which will be called $f(scum)_{\theta}$. Particularly, n expressed just in terms of G, will be equal to the inverse of G itself, because the lower the G, the more equal the probability to win in each district, hence the more n. Therefore, the

[94] as defined by Taagepera (ibid. (p. 220,248)).

general cumulative distribution function expressed in terms of v, N and G is defined as follows:

$$\mathrm{f(scum)}_\theta = \int_0^\theta \frac{\mathrm{v}^{\frac{1}{G}}}{\mathrm{v}^{\frac{1}{G}} + [\mathrm{N}-1]^{1-\frac{1}{G}}(1-\mathrm{v})^{\frac{1}{G}}}$$

I can substitute N with N′ (if known, otherwise I can just use N), and v with T, as I am determining how many parties could potentially be excluded in the interval between 0 to T. Using the same logic set previously, I can write a simple proportion to find G′, to then recalculate the general law of minority attrition - in presence of a threshold - with this new G′.

$$G : 1 = G' : 1 - f(scum)_T \Rightarrow G' = G(1 - f(scum)_T)$$

I now look at the majority premium, which is another corrective appearing in some electoral systems. In order to formalize the impact of majority premium on electoral supply, I need to identify s_1, the biggest party-coalition which will benefit from the majority premium. In order to do so, I recall the formula developed in chapter three, which best explains s_1 in function of the number of parties – effective and not -:

$$\mathrm{s}_1|(\mathrm{N}_2, \mathrm{N}_0) = \left(\frac{1}{\mathrm{N}_2{}^{\alpha_1}}\right)^{\frac{\mathrm{N}_2{}^{\alpha_2}}{\mathrm{N}_0{}^{\alpha_3}}} = \left(\frac{1}{\mathrm{N}_2{}^{0.86}}\right)^{\frac{\mathrm{N}_2{}^{0.05}}{\mathrm{N}_0{}^{0.1}}}$$

Following the same logic used to formalize the impact of the threshold on electoral supply, I include considerations for G, as it is specular to the elector strategic vote, in this case measuring the strategic alliances of parties on a territorial basis: in an extreme case, when parties in the system all present the same share, G would be 0 and s_1 would be equal to the number of parties; in the opposite case, in presence of just one party, G would be 1 and s_1 would be equal to 1.

This theoretical approach needs to be balanced with empirical considerations. As $\mathrm{s}_1|(\mathrm{N}_2, \mathrm{N}_0)$ presents exponents for each element of the equation, I can apply the geometric mean to each exponent and the G relation just defined, as follows:

$$s_{1,\mathrm{G}} = \left(\frac{1}{\mathrm{N}_2{}^{\sqrt{0.86(1-G)}}}\right)^{\frac{\mathrm{N}_2{}^{\sqrt{0.05(1-G)}}}{\mathrm{N}_0{}^{\sqrt{0.1(1-G)}}}}$$

I can now enrich this approach, including the coefficient of dis-proportionality n (rearranged in the domain [0,1]), to also consider the other institutional components - P, S, M and G_d –. Looking at the extreme cases, N and G are the sole variables which determine s_1 for the minimum value of n equal to 1, because being the system perfectly proportional, P, S, M and G_d do not impact in any known form; for the maximum value of $n = \infty$, the biggest party will obtain 100% of votes (equal to 1) because the dis-proportionality due to the electoral system is maximum. Lastly, I need to subtract the term of the interaction among each other (because N and G are already into n), always remaining in domain [0,1] as required:

$$s_1 = s_{1,\mathrm{G}} + \left(1 - \frac{1}{n}\right) - s_{1,\mathrm{G}}\left(1 - \frac{1}{n}\right)$$

At this point, I can calculate the net majority premium ΔPM awarded to the biggest party, defined as the difference between the maximum share of seats (MJP, expressed from 0 to 1) awarded to the winner by the electoral law, and what would be attributed to biggest party share (s_1) without the majority premium.

$$\Delta PM = MJP - \left[s_{1,\mathrm{G}} + \left(1 - \frac{1}{n}\right) - s_{1,\mathrm{G}}\left(1 - \frac{1}{n}\right)\right]$$

For a more precise calculus I could apply the previous utility curve posing the derivative equal to 1 and selecting the maximum solution. These solutions are exchangeable. This is another useful estimate tool to calculate strategies before the elections and to do electoral engineering.

As there is a strategic element influencing voting outcomes, the gap between votes obtained by the biggest party (v_1) and majority premium determines the probability of obtaining such premium: the smaller the gap between v_1 and MJP, the higher the probability for the biggest party to be awarded the majority premium. In terms of electoral utility, this follows the opposite logic: the smaller the gap – then the smaller the potential electoral gain of v_1 – the smaller the utility for the biggest party.

Having formalized MJP, I can now proceed to calculate the probability that this is realized, based on the number of votes. I can identify three components:

1) in a scenario where all parties present the same gap between their v and MJP, electoral utility would decrease, being all parties potential competitors for the same premium. Then, the probability to win the MJP depends only on the dispersion of party shares caught by the Effective Gini index: for instance, if G is equal to 0, then all parties are equal, and the probability to win the MJP tends to zero; likewise, for G equal to 1 the majority premium would be meaningless because one party already has all votes and therefore all seats.

2) The degree of dis-proportionality n of the utility curve also plays a part in determining the probability of winning the MJP: in fact, the minimum value n=1 determines a pure proportional electoral system which would result in a minimal gain for v_1 (and no other party), whereas in the opposite case, $n = \infty$ determines a highly dis-proportional system which would result in a maximum gain for v_1 potentially equal to 1.

3) A majority premium has a minimum threshold to activate itself - minp - and a maximum value after which it stops to produce effects - maxp -. Hence, I can define MJP's barycenter as $B = \frac{maxp+minp}{2}$. Applying a generalized probability density Beta function P(Beta) = $\frac{v^{\alpha-1}(1-v)^{\beta-1}}{\int_0^1 v^{\alpha-1}(1-v)^{\beta-1}\,dv}$, I know that the maximum of this function corresponds to the mean point $v = \frac{\alpha}{\alpha+\beta}$, as well as that the higher the coefficients, the lower the variance from the mean point, therefore:

$$P(MJP) = \frac{v^{(1-B)^{-1}\frac{1}{maxp-minp}-1}(1-v)^{B^{-1}\frac{1}{maxp-minp}-1}}{\int_0^1 v^{(1-B)^{-1}\frac{1}{maxp-minp}-1}(1-v)^{B^{-1}\frac{1}{maxp-minp}-1}\,dv}$$

At this point I can formulate a comprehensive probability function that considers all these components, having a domain [0,1]. Being this a pure statistical function in which all components have the same impact, I sum all of them; moreover, in order to maintain a domain [0,1] for each v, I include the subtraction of the product of these three components multiplied by two, obtaining:

$$P(\Delta\mathrm{PM}) = 4G(1-G) + \left(1-\frac{1}{n}\right) + P(MJP) - 2 * 4G(1-G)\left(1-\frac{1}{n}\right)P(MJP)$$

To know whether the introduction of an MJP is beneficial, considering the variables here discussed, I need to formulate an inequality imposing that the ΔPM for its probability to be attributed to the winning party must be greater than or equal to the ΔPM in the complementary case in which the other parties take the MJP which is therefore equally distributed among the other parties. This is a purely theoretical consideration as, in case of a loss for the biggest party, the MJP would practically be assigned to the second party; however, it is necessary to capitalize the interests of all other parties in the system.

Practically, I start from Taagepera's formula to quantify the remaining parties (2007b, p. 156) and I re-elaborate the average of the remaining parties in function of the Effective number of parties as follows:

$$p_{n-s_1} = \frac{\left(1 - p_1\right)}{N^{\frac{3}{2}} - 1}$$

Therefore, in order to calculate ΔPM using the formula defined above, in function of $s_{1,\mathrm{G}}$, I can better express ΔPM in function of a refined variable $s_{\mathrm{n}-s_{1,\mathrm{G}},\mathrm{G}}$ as follows:

$$s_{\mathrm{n}-s_{1,\mathrm{G}},\mathrm{G}} = \frac{\left(1 - s_{1,\mathrm{G}}\right)}{N^{\frac{3}{2}} - 1}$$

Hence, the inequality expressed above, will be formulated as:

$$\Delta\mathrm{PM} * \mathrm{P}(\Delta\mathrm{PM}) \geq \Delta\mathrm{PM}_{s_{\mathrm{n}-s_{1,\mathrm{G}},\mathrm{G}}}[1 - \mathrm{P}(\Delta\mathrm{PM})]$$

9.4. Squaring the circle: the party, the electors and the electoral general equilibrium

At this point, I have all the tools to calculate the electoral general equilibrium for any generalized electoral competition. This equilibrium point is determined by the crossing of the electoral supply conditions formalized in the previous sections with voters' electoral demand explored below.

All the tools provided so far can support the biggest party in maximizing its own electoral profit, for example, by evaluating: the opportunity of proposing a certain electoral reform (determining the width of districts M, a hypothetical majority premium MJP, a certain T) and/or an electoral alliance (modifying G_d, impacting on n and changing the curve of utility - the law of minority attrition -).

After considering all these strategies and counter-strategies, the gain to s_1 can generally be calculated, knowing v_1, using the previous relation $s_1 = s_{1,\mathrm{G}} + \left(1 - \frac{1}{n}\right) - s_{1,\mathrm{G}}\left(1 - \frac{1}{n}\right)$ and the formula relative to the MJP – if needed -. However, a more precise theoretic generalized calculus can be produced from the above cumulative party function. I impose the derivative equal to 1 and select the maximum solution of s, since from this point any marginal increase ε of v would reduce the gain $(s_1 - \mathrm{v}_1)$ for any $\mathrm{v} + \varepsilon$. Substituting the indexes G and N with G′ and N′ – previously used – to represent the final function after any eventual changes, I obtain:

$$v_1 = \max\left[\frac{\mathrm{d}}{\mathrm{dv}}\left(\frac{\mathrm{v}^{\frac{1}{G'}}}{\mathrm{v}^{\frac{1}{G'}} + [\mathrm{N}' - 1]^{1-\frac{1}{G'}}(1-\mathrm{v})^{\frac{1}{G'}}}\right) = 1\right] = \max[\mathrm{dLMA} = 1]$$

The formula above means that even the largest party has an interest in optimizing the votes obtained, and not just simply getting the maximum votes possible (at least in a democratic and competitive system). With reference to the graph in figure 54, when the derivative is equal to 1, it means that in that point the infinitesimal of the delta seats (*ds*) is equal to the delta votes (*dv*). In this case dv represents the marginal costs, and ds the marginal utilities. Having defined for simplicity the derivative of the modified law of minority attrition as *dLMA* - it follows that:

$$max\,[dLMA \;=\; 1] \Longrightarrow\; dv \;=\; ds$$

Note that max has been entered for *dLMA* in order to select the greater solution between the two existing ones. In economic terms, a party ranks in max [*dLMA* = 1] rather than equal to 0, due to its particular form deriving from the utility function. In fact, in this point, any action aimed at increasing votes would be inefficient as the party would already hold an absolute majority of seats and benefited from the entire *LMA* segment - where $ds > dv$, which starts from the inflection point at the abscissa $v = \frac{1}{N'}$, and arrives precisely at max (*dLMA* = 1), since ($max\;[dLMA\; =$

$1]) < v < 1 \Rightarrow dv < ds$.

The next step is to substitute v_1 as found in the general utility function, obtaining s_1. However, the previous formula (in n, N and G) could be a good approximation, as a handy trade off which could well estimate the value s_1. In presence of the MJP, it would be better to use the sum of the two members of the MJP equilibrium equation, obtaining the expected value of $s_1 = \Delta\mathrm{PM} * \mathrm{P}(\Delta\mathrm{PM}) + \Delta\mathrm{PM}_{s_{\mathrm{n}-s_{1,\mathrm{G}},\mathrm{G}}}[1 - \mathrm{P}(\Delta\mathrm{PM})]$.

Hence, having the value of s_1 for any case, I can finally consider the probability of the biggest party benefiting from any of the above-mentioned electoral correctives – also jointly – using the same set of formulas used for the MJP, then:

$$(s_1 - \mathrm{v}_1) * \mathrm{P}(v_1) \geq \frac{(s_1 - \mathrm{v}_1)}{N^{\frac{3}{2}} - 1}[1 - \mathrm{P}(v_1)]$$

This formula represents just the opportunity cost for parties; I now need the same for electors. The opportunity cost for electors can be represented by the strategic vote, as this means the insincere preference expressed by electors, as I have previously introduced. It is proportional to the dis-proportionality generated by the system, which can be simplified as $\left(1 - \frac{1}{n}\right)$, in the domain [0,1], because electors have the propensity to more likely change their preference if they know that voting for a bigger party implies a heavier weight of their vote.

This concept is however conditional to the fact that the party system has a particular concentration. Firstly, I assume the possibility to measure the G index for the "as sincere as possible" vote preferences (G_s). These could be expressed when surveying electors on their propensity to vote parties, to obtain a simultaneous estimate of an expression of both a pure proportional vote and a general majoritarian one, as done for the Italian general elections (from 1994 to 2001) in Figure 53, or for the more recent local elections (Gschwend Stoiber, 2014).

More in detail, for G_s equal to 0 electors would have 0 interest in changing their preference because no party is a favorite winner and all parties compete starting with the same expectation to be the winner; likewise, on the opposite case, for G_s equal to 1, electors would not change their preference because one party has all the consensus and it is guaranteed to win. Between these two extremes, voters get the maximum utility from supporting a candidate which they expect to win (the electoral demand), instead of another party which is the most preferred but with less chances to win.

Nevertheless, the utility value must be net of electoral supply, particularly if the party concentration in the districts-colleges is very high: the high concentration could happen either because parties decided to coalize or simply because a party's or coalition's dominant position exists on average, or both; these would determine the inability for voters to change the results by changing their preferences, since the final outcomes would be already determined.

Therefore, the proxy for the electoral demand is given by the difference 1-$(G_d - G)$; I subtract G because I consider electoral demand net of parties' strategies at the national level.

$$\text{SV} = \left(1 - \frac{1}{n}\right) * \left(4G_s(1 - G_s)\right) * \left(1 - (G_d - G)\right)$$

I am now able to build the general equilibrium, putting to system two inequalities and one equation:

1) the first inequality represents the electoral supply, which is the final formulation obtained from the law of minority attrition in consideration of all correctives and conditions examined in sections 9.2 and 9.3; this inequality sets the gain for the biggest party or coalition and the probability to become the winner, greater than or equal to the eventual cost of being one of the losers.

2) The second inequality sets the gain for the biggest party or coalition and the probability to become the winner, less than or equal to the strategic vote; in fact, the latter represents a counter-strategy, or a cost that electors bear to compensate electoral system and party strategies overall. This inequality links electoral supply and demand.

3) The equation represents the degree of democracy of the political system, inversely proportional to R, in [0,1]. This equation links strategic vote with the votes gained by the biggest party, in function of the ideology applied.

There is an ideological component to consider when looking to optimize an entire political system. On one hand, utilitarians will not consider a dis-representation factor because they believe that the dis-representation produced in disfavor of the remaining parties - equal to $(s_1 - v_1)$ - has been compensated by the gain of "governability" by the first party; in this case, the dummy variable R in the model below would be equal to 0. On the other hand, the Rawlsian theory (1971) considers $(s_1 - v_1)$ costs. In fact, because of the *veil of ignorance*, electors of a small party might be damaged (if not now, in the future); along the same lines, Popper considers minority protection a warranty for democracy (1996); therefore, the R considering this Rawlsian "Maximin" theory would be equal to 1.

In reality, except for R equal to 0, electors will always be subject to a "voter paradox", representing the impossibility of maximization of electors' utility overall, as Arrow theorized (1951). Therefore, in order to make democracy work, I select a value for R in the middle of its domain, for example using the "quadratic social welfare function", in this case equal to $R = SV * (s_1 - v_1)$: this would give more weight to Rawls' principle, if strategic vote happens jointly with the dis-proportionality in function of the biggest party.

The extreme case presents $(s_1 - v_1)$=1 and $SV = 1$, meaning that no elector voted sincerely and the biggest party won all seats with almost zero votes: this surely does not represent a democratic system, and for this reason the minority warranty element will become top priority – assert which all can agree on, if a democratic system is to be guaranteed - then the Rawls' criterion assumes 100% of importance, implying an R=1.

In all other combinations, the model operates in a more or less democratic system, until R reaches 0, where all electors either express totally sincere preferences or we are in a perfectly proportional system, with no strategic vote, or both, and where Rawls' criterion of justice would be meaningless since the system is already perfectly equalitarian. A representation of the pay offs of the "quadratic social welfare function": $R = SV * (s_1 - v_1)$ is in table 32 below:

Table 32 Pay offs of the "quadratic social welfare function" $R = SV * (s_1 - v_1)$.

$s_1 - v_1$	SV	Pay offs
0	0	(0,0) system perfect egalitarian-> win utilitarianism
0	1	(0,1)
1	0	(1,0)
1	1	(1,1) Rawls wins: not democratic regime

Here is the representation of the general equilibrium system with the two inequalities and one equation explained above.

$$\begin{cases} (s_1 - v_1)[1 - P(v_1)] \leq (s_1 - v_1)P(v_1) \\ (s_1 - v_1) \geq SV + (s_1 - v_1)R - SV(s_1 - v_1)R \\ R = SV * (s_1 - v_1) \end{cases}$$

10. Conclusions and Discussion

In this last chapter, I would like to show an overview of all the new tools and knowledge, their reciprocal links, their use and their added value.

This thesis offers a wide variety of tools from extremely specific to more theoretical, respectively allowing the possibility to find quick and concrete solutions to political sciences problems and providing general tools to widely extend this new knowledge to all social sciences. As Antiseri remarks, basic research is vital in itself (2007) bringing to unintentional consequences in the knowledge. This is the base for new theories. Nevertheless, even the more theoretical approach shown can be easily applied to concrete scenarios simply by substitution of the formulas here presented; in fact, sometimes, simplified formulations are presented, such as in chapter eight and nine, allowing for a flexible application in function of the contexts, as I will recap below.

I could summarize that the final aim of this thesis is the normative building of an optimal electoral system, which can warrant both logical coherence and social equity as categorized by Arrow (1951). The electoral system represents - in fact - the more quickly modifiable part of political systems, in function of the given – endogenous - political and cultural variables, such as N_2, N_0, G, SV, R, but also of the unmodifiable institutional variable P.

Therefore, for each given political system, optimizations can be performed by embracing the so-called "gradualistic engineering" approach, as a trial-by-error process that must follow the democracy, as it happens in a pure science (Id. (2007, p. 520-1)). In concrete, an optimal electoral reform, applied in a context of political culture in a time t, will produce some rebalancing of itself in the time t+1, simply because society changed over time; for these reasons, the electoral system should be adaptive. Then, politics follow a trial-by-error process trying to find their stabilization, as a one-shot solution does not exist that is always valid, also according to Hayek's principle of exploration of the unknown and error correction (1982). The parallelism is similar to Dahl's reasoning - given by the homonymous "Dahl's box" - regarding the transition of political regimes, which must be gradual for a durable democratic transaction (1971).

Then, applying interdisciplinary knowledge from political science, probability, mathematics, physics and economics, it will be possible to find the general electoral equilibrium, as shown in chapter nine, taking into account $s_1$and v_1, SV, R and the correctives of electoral systems (block threshold and majority premium). Like a waterfall, all chapters are connected using substitutions, allowing "the connection among connections" (Taagepera R. , 2015, p. 89-95).

In particular, in chapter nine, SV is obtained as a function of G and n; in turn, it is possible to derive n by N, P, S, M, T, MJP and G, as shown in chapter two; furthermore, M and S are optimized in chapter eight thanks to differential calculus; the leftovers (N and P) – as said – are endogenous. Moreover, chapter two has deeply analyzed the majority premium, thresholds and the other majority correctives, then going to precociously expand the expression $P(v_1)$ necessary to optimize the final equations of chapter nine, which also considers the other variables s_1, v_1, SV, and R. I am going to explain these connections, step by step.

Concerning s_1 used in chapter nine, chapter three can be applied, mainly if the aim is to evaluate the impact of any new electoral rule applied in a specific context or "created in a laboratory". v_1 can also be derived starting from s_1 predicted in chapter two, by means of the relations formalized by Taagepera and Shugart (2017, p. 144, formula 9.1). In this case, and more in general, this thesis is integrable for any possible substitution, firstly with the most recent publication Votes from Seats (Shugart - Taagepera, 2017) and secondly with the Predicting Party Sizes (Taagepera R. , 2007b).

Continuing the links among the thesis, I review another important optimization at the level of ideological competition. This is a crucial cleavage that determines electoral dynamics and competition among parties. The strategic vote introduced in chapter nine has revealed that it modifies votes for all parties, following this logic: the more the presence of majoritarian correctives in the electoral system to the advantage of the biggest party, the more n - and the more dis-representation -, the more SV in favor of the biggest party.

Chapter six goes to create a simplified final three-variate model able to connect the positional competition (the summation of the ideological areas occupied exclusively by one party) with N and n. It allows to recreate the ideological Beta functions for each ideal-typical N party in the system in a probabilistic way. By substituting another relation also found in chapter six, it connects N, positional competition, n and the px dimension (average ideological distance among the parties). Chapter five is indispensable to chapter six because of the Beta functions, chapter six is linked to chapter two through the n calculus, and to chapter seven because the latter allows expanding the dataset by means of predictions of the positional data and the electoral flows of the swing votes from an election to the next one.

Then, the theoretical models of chapter eight could be applied on an empirical basis to find optimal n values and their relative optimal positional party competition. In the same way, chapter nine can take advantage of the same application, to find, for example, the optimal SV, which implies the optimal R, allowing to measure the equity degree of a specific democratic system, given a particular party system.
Even though until now N is assumed as endogenous, it could also be optimized, if required, as shown in chapter eight, or by means of the relation "best fit" of D_2 expressed in terms of $(s_1 - v_1)$ given by Shugart and Taagepera (2017, p. 143)[95], depending on the situation.

Chapter four (the multi-pick – up to three maximums - probability density function) could be applied for an in-depth calculus of the analysis in both chapter two and chapter nine, also taking advantage of chapter three formulas by substitution; in this way, the party system can be described with more precision. A multi-pick function will certainly better measure the effects of electoral correctives such as majority premiums, threshold and others, not only because of the multi-pick but also because the feature of this function is probabilistic. A perspective application of this function could be to estimate the allocation of seats in the districts on a nationwide basis, by means of the function's cumulative side, in which the values of N_0 and N_2 will be manipulated in function of the differentials that they present from nationwide to district/circumscriptional level.

Here I use the methodology "connection among connection", able to minimize the error (Taagepera R. , 2015, p. 89-95; 215-219), because I am using a chain of several variables interlocked, but also because the most advanced statistical function used is able to catch up to three picks - concerning the frequency - of the party distribution. In this way, this new function presents many other potential applications, for example, for electoral simulations at national level, but also applicable to the district one. In this case, this new function uses N and N_0 with reference to the district, to predict how the national level determines the seats allotment in each district/circumscription. Thus, this application could enrich the current approach (Cfr. (Shugart - Taagepera, 2017, p. 236-258)) that focuses on the correlation between the (effective or nominal) number of parties at district/circumscription level with M.

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[95] which has an $R^2 = 91.3\%$.

Another frontier for the thesis' application could be represented by an implemented Cabinet Life estimation. A measuring of the parties' blackmail and/or coalition power could be introduced, generally coming from small parties (Downs, 1957); (Fisichella, 2009, p. 241). This can be done operationally, introducing variables that modify the exponent of N, which is currently equal to two in the formula of cabinet duration, and finally resulting in a more comprehensive model of cabinet duration. I capitalize on important information about the whole party distribution, considering the standardized measure of skewness [0,1], calculated in chapter four and for which it is useful to consider: 1) the Feld and Grofman's variance (2007), 2) the prediction of each party share - from N - given by Taagepera (Taagepera R. , 2007, p. 156-7), which I integrate through some adjustments that come from the impact of N_2, M and S on s_1, and 3) the effective Gini index.

In conclusion, I passionately believe that all the tools introduced in this thesis can form a broad and interdisciplinary platform of contents and methodologies that can expand the knowledge in political science and generally in social sciences. For example, a vital but underestimated powerful tool in political and social sciences is given by the application of LQMs to political economy estimates (Taagepera R. , 2015, p. 224-8) and in particular by time series, even though, for instance, Bernhard and Leblang (2006) have offered a hybrid approach between politics, finance and the use of time series. Social and more political scientists rarely apply nonlinear LQMs to predict future trends, unlike done by Taagepera in an elaborated way regarding a demographic model (Id. (p.222-3)). Lastly, unfortunately in political sciences the differential calculus (Id. (p.167-79)) used, in this dissertation, in a comprehensive multivariate model (chapter eight) is less diffused, even if it is widely diffused in physics and economics.

Hence huge is its potential, and that of all frontier science here proposed.

Appendixes

Appendix chapter 4

Here are all the passages of the derivative calculus of the law of minority attrition using the online platform https://www.derivative-calculator.net/.

$$\frac{d}{dv}\left[\frac{v^n}{v^n+(n-1)^{1-n}(1-v)^n}\right]$$

$$=\frac{\frac{d}{dv}[v^n]\cdot\left(v^n+(n-1)^{1-n}(1-v)^n\right)-v^n\cdot\frac{d}{dv}\left[v^n+(n-1)^{1-n}(1-v)^n\right]}{\left(v^n+(n-1)^{1-n}(1-v)^n\right)^2}$$

$$=\frac{nv^{n-1}\left(v^n+(n-1)^{1-n}(1-v)^n\right)-v^n\left(\frac{d}{dv}[v^n]+(n-1)^{1-n}\cdot\frac{d}{dv}\left[(1-v)^n\right]\right)}{\left(v^n+(n-1)^{1-n}(1-v)^n\right)^2}$$

$$=\frac{nv^{n-1}\left(v^n+(n-1)^{1-n}(1-v)^n\right)-v^n\left(nv^{n-1}+(n-1)^{1-n}n(1-v)^{n-1}\cdot\frac{d}{dv}[1-v]\right)}{\left(v^n+(n-1)^{1-n}(1-v)^n\right)^2}$$

$$=\frac{nv^{n-1}\left(v^n+(n-1)^{1-n}(1-v)^n\right)-v^n\left(nv^{n-1}+(n-1)^{1-n}n(1-v)^{n-1}\left(\frac{d}{dv}[1]-\frac{d}{dv}[v]\right)\right)}{\left(v^n+(n-1)^{1-n}(1-v)^n\right)^2}$$

$$=\frac{nv^{n-1}\left(v^n+(n-1)^{1-n}(1-v)^n\right)-v^n\left(nv^{n-1}+(n-1)^{1-n}n(1-v)^{n-1}(0-1)\right)}{\left(v^n+(n-1)^{1-n}(1-v)^n\right)^2}$$

$$=\frac{nv^{n-1}\left(v^n+(n-1)^{1-n}(1-v)^n\right)-v^n\left(nv^{n-1}-(n-1)^{1-n}n(1-v)^{n-1}\right)}{\left(v^n+(n-1)^{1-n}(1-v)^n\right)^2}$$

Alternative result:

$$=\frac{nv^{n-1}}{v^n+(n-1)^{1-n}(1-v)^n}-\frac{v^n\left(nv^{n-1}-(n-1)^{1-n}n(1-v)^{n-1}\right)}{\left(v^n+(n-1)^{1-n}(1-v)^n\right)^2}$$

Simplify/rewrite:

$$-\frac{(n-1)^{n+1}n(1-v)^n v^{n-1}}{(v-1)\left((n-1)^n v^n+(n-1)(1-v)^n\right)^2}$$

Appendixes chapter 6

As already introduced in chapter six, the appendixes of this chapter, even though based on a small sample (16 cases), can be interpreted as provisional directional correlations, and useful to draft a comprehensive model which grafts the political and institutional variables.

Appendix 6.1

The correlation between weighted ideological distance (px) and weighted Effective Number of Parties (N_{po}) is shown below. The regression has a significance of at least 99% for both the variable and the constant, the $R^2 adj.$ is 63.3% and the estimation error (Root MSE) is only 5%, as shown in Table 33 below.

Table 33 Correlation between weighted ideological distance (px) and the weighted Effective Number of Parties (N_{po})

VARIABLES	Weighted ideological distance (px)
N_{po}	0.0478 ***
	(0.00922)
Constant	0.380 ***
	(0.0372)
Observations	16
R^2	0.657
R^2Adj.	0.633
Prob > F test	0.000
Root MSE	0.0502

Standard Errors in Parentheses
p < 0.01, * * p < 0.05, * p < 0.1

Appendix 6.2

Table 34 Correlations of the n expressed by positional (px), positional on weighted N (px/N_{po}) and weighted positional competition (pxq)

VARIABLES	Model 1 px/N_{po}	Model 2 px	Model 3 pxq
n	-0.0179	-0.0494	-0.0443
	(0.0208)	(0.0408)	(0.0472)
Constant	0.189 ***	0.641 ***	0.592***
	(0.0347)	(0.0683)	(0.0790)
Observations	16	16	16
R^2	0.051	0.095	0.059
R^2 Adj.	-0.017	0.030	-0.008
Prob > F test	0.403	0.247	0.364

Standard Errors in Parentheses
p < 0.01, * * p < 0.05, * p < 0.1

Appendix 6.3

Table 35 below presents the combinations of the previous variables in function of n as dependent variable.

Table 35 Correlation models expressing n in terms of positional competition variables

VARIABLES	Model 1 n	Model 2 n	Model 3 n	Model 4 n	Model 5 n	Model 6 n	Model 7 n	Model 8 n	Beta
px	-2.550 (7.225)		-3.028 * (1.598)	-4.085 ** (1.566)				-9.008*** (2.577)	17,90%
px/ N_{po}		5.750 (18.69)					30.29*** (9.418)		
Quadratic Prox. (pxq)					-3.047* (1.467)	-7.638*** (2.500)			
pos. comp./ N_{po}				-3,736 ** (1.425)	-3.402** (1.513)	-17.42** (6.665)	19.53** (6.418)	-21.55** (8.155)	47,06%
px*pos. comp./ N_{po}		-12.28 (29.99)	-7.021 (5.405)					32.26** (14.66)	35,04%
pos. comp.	-1.231 (7.020)	1.098 (4.663)	0.290 (1.177)						
px*pos. comp.	0.375 (12.53)								
px*pos. comp./ ${N_{po}}^2$							-139.2*** (39.80)		
pxq*pos. comp./ N_{po}						26.60* (12.38)			
Constant	3.610 (4.086)	1.198 (2.823)	3.785 *** (0.986)	4.533 *** (1.035)	3.770*** (0.933)	6.300*** (1.439)	-2.387* (1.294)	7.368*** (1.546)	
Observations	16	16	16	16	16	16	16	16	
R^2	0.248	0.144	0.338	0.408	0.323	0.511	0.552	0.583	
R^2 Adj.	0.060	-0.071	0.173	0.317	0.219	0.388	0.440	0.479	
Prob > F test	0.315	0.586	0.161	0.033	0.080	0.031	0.019	0.012	
Root MSE	0.5116	0.5458	0.4697	0.4269	0.4565	0.4038	0.3947	0.3808	

Standard Errors in Parentheses
p < 0.01, * * p < 0.05, * p < 0.1

Appendix 6.4

Table 36 n expressed in function of the best model's predicted values, coming from: weighted ideological distance (px), positional party competition (pos. comp.) and the weighted Effective number of parties (N_{po}).

VARIABLES	Predicted values (px, Pos.comp. and N_{po})
n	0.583***
	(0.132)
Constant	0.667***
	(0.221)
Observations	16
R^2	0.583
R^2 Adj.	0.553
Prob > F test	0.001
Root MSE	0.2692

Standard Errors in Parentheses
***p < 0.01, * * p < 0.05, * p < 0.1

Appendixes chapter 8

Below is the derivative calculus of the differential equations created in chapter 8. I used the online platform https://www.derivative-calculator.net/.

Appendix 8.1

$$\frac{\mathrm{d}}{\mathrm{d}S}\left[\frac{2P}{\sqrt{NS}}+\frac{NS}{2}\right]$$

$$=2P\cdot\frac{\mathrm{d}}{\mathrm{d}S}\left[\frac{1}{\sqrt{NS}}\right]+\frac{N}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}S}[S]$$

$$=2P\left(-\frac{1}{2}\right)(NS)^{-\frac{1}{2}-1}\cdot\frac{\mathrm{d}}{\mathrm{d}S}[NS]+\frac{1N}{2}$$

$$=\frac{N}{2}-\frac{P\cdot N\cdot\frac{\mathrm{d}}{\mathrm{d}S}[S]}{(NS)^{\frac{3}{2}}}$$

$$=\frac{N}{2}-\frac{PN\cdot 1}{(NS)^{\frac{3}{2}}}$$

$$=\frac{N}{2}-\frac{NP}{(NS)^{\frac{3}{2}}}$$

Knowing that the derivative must be equal to 0, I can write the last equation as follows: $\frac{N}{2}=\frac{NP}{\sqrt[2]{(NS)^3}}$; solving by S, the real optimized result will be: $S^*=\frac{\sqrt[3]{4P^2}}{N}$.

Appendix 8.2

Below I show the derivative calculus to find the optimal S in its first new formulation.

$$\frac{\mathrm{d}}{\mathrm{d}S}\left[\frac{(NS)^d}{2^d}+\frac{2Pd}{\sqrt{NS}}+d-1\right]$$

$$=\frac{1}{2^d}\cdot\frac{\mathrm{d}}{\mathrm{d}S}\left[(NS)^d\right]+2Pd\cdot\frac{\mathrm{d}}{\mathrm{d}S}\left[\frac{1}{\sqrt{NS}}\right]+\frac{\mathrm{d}}{\mathrm{d}S}[d]+\frac{\mathrm{d}}{\mathrm{d}S}[-1]$$

$$=\frac{d(NS)^{d-1}\cdot\frac{\mathrm{d}}{\mathrm{d}S}[NS]}{2^d}+2Pd\left(-\frac{1}{2}\right)(NS)^{-\frac{1}{2}-1}\cdot\frac{\mathrm{d}}{\mathrm{d}S}[NS]+0+0$$

$$=\frac{d(NS)^{d-1}\cdot N\cdot\frac{\mathrm{d}}{\mathrm{d}S}[S]}{2^d}-\frac{Pd\cdot N\cdot\frac{\mathrm{d}}{\mathrm{d}S}[S]}{(NS)^{\frac{3}{2}}}$$

$$=\frac{dN(NS)^{d-1}\cdot 1}{2^d}-\frac{PNd\cdot 1}{(NS)^{\frac{3}{2}}}$$

$$=\frac{Nd(NS)^{d-1}}{2^d}-\frac{NPd}{(NS)^{\frac{3}{2}}}$$

Rewrite/simplify:

$$=\frac{d(NS)^d}{2^dS}-\frac{NPd}{(NS)^{\frac{3}{2}}}$$

Knowing that the derivative must be equal to 0, I can write the last equation as follows: $\frac{d(NS)^d}{2^dS}=\frac{NPd}{\sqrt[2]{(NS)^3}}$; solving by S, the real optimized result will be: $S^*=\frac{P^{\frac{2}{3d}}}{N}$.

Appendix 8.3

Below I show the derivative calculus to find the optimal S in its final formulation.

$$\frac{\mathrm{d}}{\mathrm{d}S}\left[\frac{2P}{(N^{e}S^{1-e})^{d}}+\frac{(N^{2e}S^{2-2e})^{d}}{2^{d}}\right]$$

$$= 2P\cdot\frac{\mathrm{d}}{\mathrm{d}S}\left[\frac{1}{(N^{e}S^{1-e})^{d}}\right]+\frac{1}{2^{d}}\cdot\frac{\mathrm{d}}{\mathrm{d}S}\left[(N^{2e}S^{2-2e})^{d}\right]$$

$$= 2P\,(-d)\,(N^{e}S^{1-e})^{-d-1}\cdot\frac{\mathrm{d}}{\mathrm{d}S}\left[N^{e}S^{1-e}\right]+\frac{d(N^{2e}S^{2-2e})^{d-1}\cdot\frac{\mathrm{d}}{\mathrm{d}S}\left[N^{2e}S^{2-2e}\right]}{2^{d}}$$

$$= -2P(N^{e}S^{1-e})^{-d-1}d\cdot N^{e}\cdot\frac{\mathrm{d}}{\mathrm{d}S}\left[S^{1-e}\right]+\frac{d(N^{2e}S^{2-2e})^{d-1}\cdot N^{2e}\cdot\frac{\mathrm{d}}{\mathrm{d}S}\left[S^{2-2e}\right]}{2^{d}}$$

$$= -2P(N^{e}S^{1-e})^{-d-1}N^{e}d\,(1-e)\,S^{-e}+\frac{d(N^{2e}S^{2-2e})^{d-1}N^{2e}\,(2-2e)\,S^{1-2e}}{2^{d}}$$

$$= \frac{(2-2e)\,N^{2e}dS^{1-2e}(N^{2e}S^{2-2e})^{d-1}}{2^{d}}-\frac{2\,(1-e)\,N^{e}Pd(N^{e}S^{1-e})^{-d-1}}{S^{e}}$$

Rewrite/simplify:

$$= \frac{(2-2e)\,d(N^{2e}S^{2-2e})^{d}}{2^{d}S}-\frac{2\,(1-e)\,Pd}{S(N^{e}S^{1-e})^{d}}$$

Knowing that the derivative must be equal to 0, I can write the last equation as follows: $\frac{(2-2e)d(N^{2e}S^{2-2e})^{d}}{2^{d}S}=\frac{2(1-e)Pd}{S(N^{e}S^{1-e})^{d}}$; solving by S the real optimized result will be: $S^{*}=\left(\frac{2^{(-d)}N^{(3de)}}{P}\right)^{\frac{1}{3d(e-1)}}$.

Appendix 8.4

Below I show the derivative calculus to find the optimal M. S^*appears simply as S just aiming to simply the notation.

$$\frac{d}{dM}\left[\frac{\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^d}{2^d}+\frac{2P}{\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^d}\right]$$

$$=\frac{1}{2^d}\cdot\frac{d}{dM}\left[\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^d\right]+2P\cdot\frac{d}{dM}\left[\frac{1}{\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^d}\right]$$

$$=\frac{d\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^{d-1}\cdot\frac{d}{dM}\left[S^{2-2e}(SM)^{\frac{e}{3}}\right]}{2^d}+2P\,(-d)\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^{-d-1}\cdot\frac{d}{dM}\left[S^{1-e}(SM)^{\frac{e}{6}}\right]$$

$$=\frac{d\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^{d-1}\cdot S^{2-2e}\cdot\frac{d}{dM}\left[(SM)^{\frac{e}{3}}\right]}{2^d}-2P\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^{-d-1}d\cdot S^{1-e}\cdot\frac{d}{dM}\left[(SM)^{\frac{e}{6}}\right]$$

$$=\frac{d\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^{d-1}\cdot\frac{e}{3}S^{2-2e}(SM)^{\frac{e}{3}-1}\cdot\frac{d}{dM}[SM]}{2^d}-2P\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^{-d-1}d\cdot\frac{e}{6}S^{1-e}(SM)^{\frac{e}{6}-1}\cdot\frac{d}{dM}[SM]$$

$$=\frac{ed\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^{d-1}S^{2-2e}(SM)^{\frac{e}{3}-1}\cdot S\cdot\frac{d}{dM}[M]}{3\cdot 2^d}-\frac{eP\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^{-d-1}dS^{1-e}(SM)^{\frac{e}{6}-1}\cdot S\cdot\frac{d}{dM}[M]}{3}$$

$$=\frac{edS^{3-2e}\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^{d-1}(SM)^{\frac{e}{3}-1}\cdot 1}{3\cdot 2^d}-\frac{eP\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^{-d-1}S^{2-e}d(SM)^{\frac{e}{6}-1}\cdot 1}{3}$$

$$=\frac{eS^{3-2e}d(SM)^{\frac{e}{3}-1}\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^{d-1}}{3\cdot 2^d}-\frac{ePS^{2-e}d(SM)^{\frac{e}{6}-1}\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^{-d-1}}{3}$$

Rewrite/simplify:

$$=\frac{ed\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^d}{3\cdot 2^d M}-\frac{ePd}{3M\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^d}$$

Knowing that the derivative must be equal to 0, I can write the last equation as follows: $\frac{ed\left(S^{2-2e}(SM)^{\frac{e}{3}}\right)^d}{3*2^d M}=\frac{ePd}{3M\left(S^{1-e}(SM)^{\frac{e}{6}}\right)^d}$; solving by M the real optimized result will be: $M^*=2^{(2/e)}S^{*(5-6/e)}P^{(2/(de))}$.

Appendix 8.5

Below I show the derivative calculus to find the optimal D_E in its first formulation. $S^*$and M^*appear simply as S and M, just aiming to simply the notations.

$$\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[\frac{k\left(1-\left(1-\frac{1}{N_{\mathrm{v}}}\right)\left(-\frac{2D_{\mathrm{E}}}{\sqrt[3]{M}\sqrt[3]{S}}+2D_{\mathrm{E}}+\frac{1}{\sqrt[3]{M}\sqrt[3]{S}}\right)\right)}{N_{\mathrm{v}}}+\frac{k}{N_{\mathrm{s}}^2}\right]$$

$$=\frac{k}{N_{\mathrm{v}}}\left(\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[-\left(1-\frac{1}{N_{\mathrm{v}}}\right)\left(-\frac{2D_{\mathrm{E}}}{\sqrt[3]{M}\sqrt[3]{S}}+2D_{\mathrm{E}}+\frac{1}{\sqrt[3]{M}\sqrt[3]{S}}\right)\right]+\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}[1]\right)+\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[\frac{k}{N_{\mathrm{s}}^2}\right]$$

$$=\frac{\left(\left(\frac{1}{N_{\mathrm{v}}}-1\right)\left(\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[-\frac{2D_{\mathrm{E}}}{\sqrt[3]{M}\sqrt[3]{S}}\right]+\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}[2D_{\mathrm{E}}]+\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[\frac{1}{\sqrt[3]{M}\sqrt[3]{S}}\right]\right)+0\right)k}{N_{\mathrm{v}}}+0$$

$$=\frac{\left(\left(-\frac{2}{\sqrt[3]{M}\sqrt[3]{S}}\cdot\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}[D_{\mathrm{E}}]\right)+2\cdot\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}[D_{\mathrm{E}}]+0\right)\left(\frac{1}{N_{\mathrm{v}}}-1\right)k}{N_{\mathrm{v}}}$$

$$=\frac{\left(-\frac{2\cdot 1}{\sqrt[3]{M}\sqrt[3]{S}}+2\cdot 1\right)\left(\frac{1}{N_{\mathrm{v}}}-1\right)k}{N_{\mathrm{v}}}$$

$$=\frac{\left(\frac{1}{N_{\mathrm{v}}}-1\right)\left(2-\frac{2}{\sqrt[3]{M}\sqrt[3]{S}}\right)k}{N_{\mathrm{v}}}$$

Appendix 8.6

Below I show the derivative calculus to find the optimal D_E in its second formulation. $S^*$and M^*appear simply as S and M, just aiming to simply the notations.

$$\frac{\mathrm{d}}{\mathrm{d}D_\mathrm{E}}\left[\frac{k\left(1-\left(1-\frac{1}{N_\mathrm{v}}\right)\left(-\frac{2D_\mathrm{E}}{\sqrt[3]{M}\sqrt[3]{S}}+2D_\mathrm{E}+\frac{1}{\sqrt[3]{M}\sqrt[3]{S}}\right)\right)}{N_\mathrm{v}}+\frac{\sqrt[3]{2}k\sqrt[3]{D_\mathrm{E}}}{\sqrt[9]{M}N_\mathrm{s}^{\frac{2}{3}}\sqrt[9]{S}}\right]$$

$$=\frac{k}{N_\mathrm{v}}\left(\frac{\mathrm{d}}{\mathrm{d}D_\mathrm{E}}\left[-\left(1-\frac{1}{N_\mathrm{v}}\right)\left(-\frac{2D_\mathrm{E}}{\sqrt[3]{M}\sqrt[3]{S}}+2D_\mathrm{E}+\frac{1}{\sqrt[3]{M}\sqrt[3]{S}}\right)\right]+\frac{\mathrm{d}}{\mathrm{d}D_\mathrm{E}}[1]\right)+\frac{\sqrt[3]{2}k}{\sqrt[9]{M}N_\mathrm{s}^{\frac{2}{3}}\sqrt[9]{S}}\cdot\frac{\mathrm{d}}{\mathrm{d}D_\mathrm{E}}\left[\sqrt[3]{D_\mathrm{E}}\right]$$

$$=\frac{\left(\left(\frac{1}{N_\mathrm{v}}-1\right)\left(\frac{\mathrm{d}}{\mathrm{d}D_\mathrm{E}}\left[-\frac{2D_\mathrm{E}}{\sqrt[3]{M}\sqrt[3]{S}}\right]+\frac{\mathrm{d}}{\mathrm{d}D_\mathrm{E}}[2D_\mathrm{E}]+\frac{\mathrm{d}}{\mathrm{d}D_\mathrm{E}}\left[\frac{1}{\sqrt[3]{M}\sqrt[3]{S}}\right]\right)+0\right)k}{N_\mathrm{v}}+\frac{\sqrt[3]{2}\cdot\frac{1}{3}D_\mathrm{E}^{\frac{1}{3}-1}k}{\sqrt[9]{M}N_\mathrm{s}^{\frac{2}{3}}\sqrt[9]{S}}$$

$$=\frac{\sqrt[3]{2}k}{3\sqrt[9]{M}N_\mathrm{s}^{\frac{2}{3}}\sqrt[9]{S}D_\mathrm{E}^{\frac{2}{3}}}+\frac{\left(\left(-\frac{2}{\sqrt[3]{M}\sqrt[3]{S}}\cdot\frac{\mathrm{d}}{\mathrm{d}D_\mathrm{E}}[D_\mathrm{E}]\right)+2\cdot\frac{\mathrm{d}}{\mathrm{d}D_\mathrm{E}}[D_\mathrm{E}]+0\right)\left(\frac{1}{N_\mathrm{v}}-1\right)k}{N_\mathrm{v}}$$

$$=\frac{\sqrt[3]{2}k}{3\sqrt[9]{M}N_\mathrm{s}^{\frac{2}{3}}\sqrt[9]{S}D_\mathrm{E}^{\frac{2}{3}}}+\frac{\left(-\frac{2\cdot1}{\sqrt[3]{M}\sqrt[3]{S}}+2\cdot1\right)\left(\frac{1}{N_\mathrm{v}}-1\right)k}{N_\mathrm{v}}$$

$$=\frac{\sqrt[3]{2}k}{3\sqrt[9]{M}N_\mathrm{s}^{\frac{2}{3}}\sqrt[9]{S}D_\mathrm{E}^{\frac{2}{3}}}+\frac{\left(\frac{1}{N_\mathrm{v}}-1\right)\left(2-\frac{2}{\sqrt[3]{M}\sqrt[3]{S}}\right)k}{N_\mathrm{v}}$$

Knowing that the derivative must be equal to 0, I can write the last equation as follows: $\frac{\sqrt[3]{2}k}{3\sqrt[9]{M}N_S^{\frac{2}{3}}\sqrt[9]{S}D_E^{\frac{2}{3}}}=-\frac{\left(\frac{1}{N_v}-1\right)\left(2-\frac{2}{\sqrt[3]{MS}}\right)k}{N_v}$; solving by D_E, and knowing that k is theoretically higher than 0 - therefore simplifiable - the real optimized result will be equal to: $D_E{}^*=\frac{N_v{}^2\sqrt[3]{M^*S^*}}{6\sqrt{3}(N_v-1)\left(\sqrt[3]{M^*S^*}-1\right)\sqrt{\frac{(N_v-1)\mathrm{N}_S{}^2\left(\sqrt[3]{M^*S^*}-1\right)}{N_v{}^2}}}$.

Appendix 8.7

Below I show the derivative calculus to find the optimal D_E in its last formulation. $S^*$and M^*appear simply as S and M, just aiming to simply the notations.

$$\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[k\left(\frac{D_{\mathrm{E}}}{\sqrt[3]{M}\sqrt[3]{S}}-D_{\mathrm{E}}-\frac{1}{2\sqrt[3]{M}\sqrt[3]{S}}+\frac{1}{2N_{\mathrm{v}}^{2}}+\frac{1}{2}\right)+\frac{\sqrt[3]{2}k\sqrt[3]{D_{\mathrm{E}}}}{\sqrt[9]{M}N_{\mathrm{s}}^{\frac{2}{3}}\sqrt[9]{S}}\right]$$

$$=k\left(\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[\frac{D_{\mathrm{E}}}{\sqrt[3]{M}\sqrt[3]{S}}\right]+\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}[-D_{\mathrm{E}}]+\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[-\frac{1}{2\sqrt[3]{M}\sqrt[3]{S}}\right]+\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[\frac{1}{2N_{\mathrm{v}}^{2}}\right]+\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[\frac{1}{2}\right]\right)+\frac{\sqrt[3]{2}k}{\sqrt[9]{M}N_{\mathrm{s}}^{\frac{2}{3}}\sqrt[9]{S}}\cdot\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}\left[\sqrt[3]{D_{\mathrm{E}}}\right]$$

$$=k\left(\frac{1}{\sqrt[3]{M}\sqrt[3]{S}}\cdot\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}[D_{\mathrm{E}}]+\left(-\frac{\mathrm{d}}{\mathrm{d}D_{\mathrm{E}}}[D_{\mathrm{E}}]\right)+0+0+0\right)+\frac{\sqrt[3]{2}\cdot\frac{1}{3}D_{\mathrm{E}}^{\frac{1}{3}-1}k}{\sqrt[9]{M}N_{\mathrm{s}}^{\frac{2}{3}}\sqrt[9]{S}}$$

$$=\frac{\sqrt[3]{2}k}{3\sqrt[9]{M}N_{\mathrm{s}}^{\frac{2}{3}}\sqrt[9]{S}D_{\mathrm{E}}^{\frac{2}{3}}}+k\left(\frac{1}{\sqrt[3]{M}\sqrt[3]{S}}-1\right)$$

Knowing that the derivative must be equal to 0, I can write the last equation as follows: $\frac{\sqrt[3]{2}k}{3\sqrt[9]{M}N_S^{\frac{2}{3}}\sqrt[9]{S}D_E^{\frac{2}{3}}}=-k(\frac{1}{\sqrt[3]{MS}}-1)$; solving by D_E, and knowing that k is theoretically higher than 0 - therefore simplifiable - the real optimized result will be equal to: $D_E{}^*=\frac{\sqrt{\frac{2}{3}}\sqrt[3]{M^*S^*}}{3\left(\sqrt[3]{M^*S^*}-1\right)\sqrt{\mathrm{N}_S{}^2\left(\sqrt[3]{M^*S^*}-1\right)}}$.

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